Change in risk

In the previous chapter we have defined the concept of risk aversion by assuming that a risk-averse decision maker refuses to take a zero mean risk. A risk-averse decision maker prefers to stay with his sure income w rather than to take any zero mean risk and play the lottery $w + \widetilde{x}$. The comparison between risk and certainty is obviously clear. But nothing has been sayed about the comparison between two zero-mean risks

In some circumstances, the comparison between two different lotteries seems clear : if $\widetilde{w}_1$ and $\widetilde{w}_2$ have the same mean and extremal events of w_1 have a higher probability and central events a lower, then intuition suggests that $\widetilde{w}_1$ is more risky than $\widetilde{w}_2$. What does this mean for the utility functions that are compatible with this ordering?

In this chapter we will give some rigorous definition to the sentence "$\widetilde{w}_1$ is dominated by $\widetilde{w}_2$ " by saying that a certain class of utility functions lead to chose $\widetilde{w}_2$ rather than $\widetilde{w}_1$.

1 Increases in risk

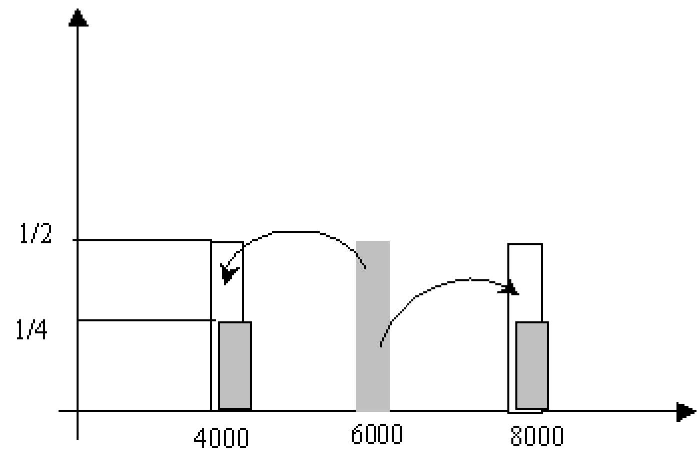
In this paragraph we retrict attention to the comparision of lotteries with the same expected outcome. We say that the risk increases between two such lotteries if all risk averse decision maker prefer the first to the second : increase in risk makes all risk averse people worse off.

Definition 1 *Let $\widetilde{w}_1$ and $\widetilde{w}_2$ two lotteries with the same expectation. We say that $\widetilde{w}_1$ is more risky than $\widetilde{w}_2$ if all risk averse decision maker prefers $\widetilde{w}_2$: $\forall$ u increasing concave function, $E(u(\widetilde{w}_1)) \leq E(u(\widetilde{w}_2))$.*

We will see in this chapter that there are several ways to characterise such changes in risk.

1.1 Mean-preserving spread

Just recall the problem of our merchant Sempronius and take the graph of the probability distributions of the wealth when he trusts one boat (white bars) and when he trusts two (grey bars). When he loads only one boat, 4000 and 8000 have equal probability (1/2). When he splits in two boats 4000 and 8000 have equal probabilities (1/4) and 6000 has the probability 1/2. The white distribution is a mean-preserving spread of the grey one in the sense that they have the same expectation (6000) and that the white is obtained by lowering the probability of 6000 (to 0) and allocating it to extrême points (4000 and 8000).



This leads to the general following definition.

Definition 2 *Given two random variables $\widetilde{w}_1$ and $\widetilde{w}_2$. Note f_1 and f_2 their density (probability distribution.) We say that $\widetilde{w}_1$ is a mean-preserving spread of $\widetilde{w}_2$ if :*

(i) $E(\widetilde{w}_1) = E(\widetilde{w}_2)$

(ii) $\exists$ $I = [\alpha, \beta]$ *interval, such that* $f_1(s) \leq f_2(s) \iff s \in [\alpha, \beta]$

In other words, the density his lower in I and higher outside, the mean being preserved.

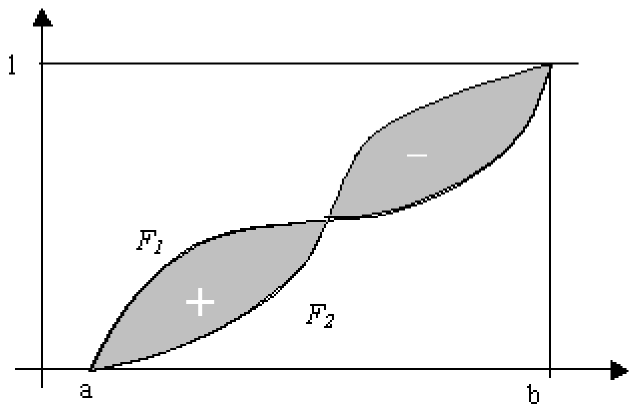
To fix ideas take random variables whose support are all in the interval $[a, b]$, and whose cumulative distributions $F_i(s) = \int_a^s f_i(w)dw$.are continuous (although the result is valid for more general (with atoms) distributions). A very simple calculus (integration by parts) gives :

$$E(\widetilde{w}_i) = a + \int_a^b (1 - F_i(s))ds = b - \int_a^b F_i(s)ds$$

Expectation is hence measured by a plus the area above the cumulative distribution.

Take the function $F_1(s) - F_2(s)$. It is nul for $s = 0$ and $s = 1$.

If $\widetilde{w}_1$ is a mean-preserving spread of $\widetilde{w}_2$, there exists $[\alpha, \beta]$, such that, for $s < \alpha$ and $s > \beta$, $F_1(s) - F_2(s)$. is strictly increasing, and for s between α and β, $F_1(s) - F_2(s)$. is decreasing Hence F_1 and F_2 cross only once in $]a, b[$. at a point c for which $F_1(c) = F_2(c)$; F_1 is above F_2



for $s \leq c$ an below for $s \geq c$. Moreover, As they have the same expectation, the area between F_1 and F_2 (measured positively iff $F_1 \geq F_2$) is nul.

1.2 Adding Noise

Another way to increase risk is to add "noise".

Definition 3 *$\widetilde{w}_1$ is more "noisy" than $\widetilde{w}_2$ if there exists a random variable $\widetilde{\varepsilon}$ such that*

$$\widetilde{w}_1 = \widetilde{w}_2 + \widetilde{\varepsilon}, \textit{with } E(\widetilde{\varepsilon}/\widetilde{w}_2) = 0$$

This definition means that each realistation of w_2 is itself affected by uncertainty.

A simple way to undestand what means "adding noise" is to remember the problem of Sempronius and suppose that the price of foreign spices is uncertain. If the boat perishes the wealth is 4000 but if he succeeds, the wealth is no more certain and is equal to 8000+$\widetilde{\varepsilon}$ with $E(\widetilde{\varepsilon}) = 0$..

It is easy to see (for discrete random variables but it is also true for continuous ones) that adding noise is equivalent to a succession of mean preserving spreads.

1.3 General condition

The third (equivalent) way to define increse in risk relies on its definition.. Recall that the definition says that a lottery is more risky than another if and only if all risk averse people agree on this ranking.

Take such an individual :

$$\begin{aligned} E(u(\widetilde{w}_1) - E(u(\widetilde{w}_2) &= \int_a^b u(w)(f_1(w) - f_2(w))dw \\ &= \int_a^b u'(w)(F_2(w) - F_1(w))dw \end{aligned}$$

Then define the "bi-cumulative" function by :

$$\mathcal{F}_i(s) = \int_a^s F_i(w)dw$$

We know that : $\mathcal{F}_i(a) = 0$ and (because both variables have the same expectation), $\mathcal{F}_1(b) = \mathcal{F}_2(b) = b - E(\widetilde{w}_i)$.

A second integration by parts gives hence :

$$E(u(\widetilde{w}_1) - E(u(\widetilde{w}_2) = \int_a^b u''(w)(\mathcal{F}_1(w) - \mathcal{F}_2(w))dw$$

Since we want that for all concave funcions u (that is for all negative functions u'') $E(u(\widetilde{w}_1) - E(u(\widetilde{w}_2) \leq 0$, we must have :

$$\forall w, \mathcal{F}_1(w) \geq \mathcal{F}_2(w)$$

In fact this condition is also sufficient. To see that it is interesting to make the following remark..

Remark 4 *The set of increasing concave functions is a convex (semi cone) subset of increasing functions. The extremal elements of this convex are the funcions τ_s : $w \to \min(w, s)$. Every concave increasing function is a positive weighted sum of those τ_s . Indeed, two simple integrations by parts give :*

$$\begin{aligned} u(w) &= u(a) + \int_a^b u'(s)1_w(s)ds \\ &= u(a) + [\min(w,s)u'(s)]_a^b + \int_a^b -u''(s)\min(w,s)ds \\ &= u(a) + wu'(b) - au'(a) + \int_a^b -u''(s)\min(w,s)ds \end{aligned}$$

To insure that all concave increasing functions agree on the ranking between $\widetilde{w}_1$ and $\widetilde{w}_2$, it is sufficient to insure it for the τ_s. Indeed if two functions agree on a ranking, then any convex combination (and even more every positive weighted combination) give the same ranking.

Since we have :

$$E\min(\widetilde{w}_i, s) = \int_a^b \min(w,s)f_i(w)dw = s - \mathcal{F}_i(s)$$

the condition $\forall w, \mathcal{F}_1(w) \geq \mathcal{F}_2(w)$ simply means that all decision makers having a min utility function agree on the ranking. As evey concave function can be obtained as positive combinations of such min functions, the result follows.

We have the following proposition :

Proposition 5 *Given two random variables $\widetilde{w}_1$ and $\widetilde{w}_2$ with the same mean. The following properties are equivalent.*

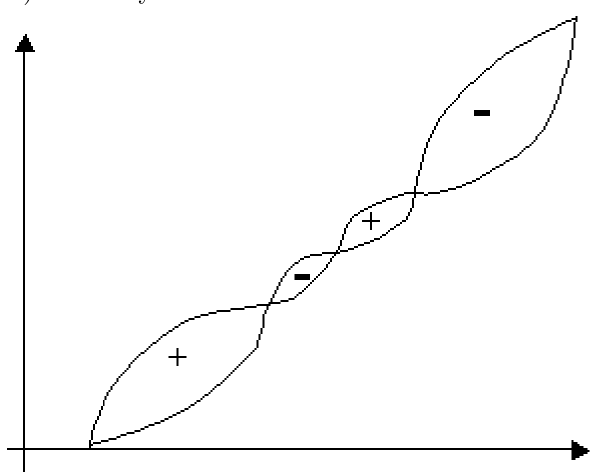
(i) *$\widetilde{w}_1$ is more risky than $\widetilde{w}_2$ (all risk averse decision maker prefer $\widetilde{w}_2$)*

(ii) *$\widetilde{w}_1$ can be obtained from $\widetilde{w}_2$ by a succession of a mean-preserving spreads.*

(iii) $\exists\widetilde{\varepsilon}$, $\widetilde{w}_1 = \widetilde{w}_2 + \widetilde{\varepsilon}$, *with* $E(\widetilde{\varepsilon}/\widetilde{w}_2) = 0$ *($\widetilde{w}_1$ is more noisy)*

(iv) $\forall w, \mathcal{F}_1(w) \geq \mathcal{F}_2(w)$

The last condition can be interpreted in the diagram of cumulative distribution. $\widetilde{w}_1$ is more risky than $\widetilde{w}_2$ if the area btween F_1 and F_2 below any level of wealth w is always positve. It is nul for $w = a$ (!) and for $w = b$ (same expectation). F_1 and F_2 can cross several times, but negative areas (that come necessarily after positive ones) are always smaller.



2 Risk and variance

What about the variance of a random variable.. Is it a good measure of risk? To show that it is not the case, take again two lotteries $\widetilde{w}_1$ and $\widetilde{w}_2$ with the same expectation. And do again the same trick :

$$\begin{aligned} var(\widetilde{w}_1) - var(\widetilde{w}_2) &= \int_a^b w^2(f_1(w) - f_2(w))dw \\ &= \int_a^b 2w(F_2(w) - F_1(w))dw \\ &= 2\int_a^b(\mathcal{F}_1(w) - \mathcal{F}_2(w))dw \end{aligned}$$

Obviously if $\widetilde{w}_1$ is more risky than $\widetilde{w}_2$ then $var(\widetilde{w}_1) \geq var(\widetilde{w}_2)$. But the converse is not true. In the risk comparision, higher moments are involved so that one can have a variable with a higher variance but for which the "increasing risk" criterion is not fulfilled.