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Financial Markets for Unknown Risks if Beliefs Differ

SUSANNE KLIMPEL
TILL REQUATE requate@mail.eco.uni-heidelberg.de
Interdisciplinary Institute for Environmental Economics, University of Heidelberg, Bergheimer Str. 20, 69115 Heidelberg, Germany

Abstract

We consider a general equilibrium model with individual and collective risks. The article builds on a contribution by Chichilnisky and Heal, who show that contingent Arrow–Debreu equilibria can also be supported in economies with Arrow securities and mutual insurance contracts. However, they show this to be true in general only if beliefs are identical, a very restrictive assumption in the context of unknown risks. Moreover, they claim complete insurance in equilibrium to be impossible if beliefs are different. We show that even with different beliefs, firstly, complete insurance is possible in each statistical state, and secondly, contingent equilibrium can still be supported in economies with insurance and securities.

Key words: individual risk, collective risk, unknown risk, financial markets, mutual insurance contracts, Arrow securities

1. Introduction

In a recent contribution, Chichilnisky and Heal [1998] study exchange economies with unknown risks in the following sense. Each household faces the risk of being in one of, say, S individual states. The risk is unknown, however, and might even be unknowable, since there is no repetition of certain events a sufficient number of times to permit estimation of their probabilities. Each household, however, has beliefs, i.e., subjective probability distributions, about being in a certain state of the world. These beliefs may differ from household to household. Such a framework is by no means unusual and is covered, in principle, by Debreu's [1959] famous Chapter 7. It is well known that in such a framework a complete set of state-contingent commodity markets would lead to a Pareto-efficient allocation. As Chichilnisky and Heal [1992] emphasize, however, "[T]his approach may be impracticable as the number of markets needed with individual risks. . . rises exponentially with the number of agents in the economy" (p. 2). The reason is that a state-contingent market approach requires decisions over the complete enumeration of all possible combinations of individuals and states over the whole population of an economy.

To reduce this complexity and to nevertheless allow for insurance of the individuals against the risks of being in certain states of the world, Chichilnisky and Heal introduce an institutional framework with two types of instruments: *mutual insurance contracts*, on the one hand, and *Arrow securities*, as introduced in Arrow [1953], on the other. To understand the role of these instruments, it is necessary to consider the possible risks in more detail. We will see that among the huge number of combinations of households and states, there are several combinations that can be considered to be statistically equivalent.

For example, consider an economy with ten agents, where each agent may be in only two different states, namely, sick or healthy. Then the economy on the whole may be in 2^{10} different states. Assume for a moment that an insurance company is going to insure all ten of these people against the financial burden in case of becoming sick (the doctors' bills). Imagine that exactly three people become sick. Then it does not matter to the insurance company which three among the ten people are sick. Thus, to the insurance company, the $\binom{10}{3}$ states of the economy in which exactly three people are sick are equivalent. If by experience the insurance company knew that on average 30% of the people became sick, i.e., if the risk were known, it could calculate premia and completely insure the people (i.e., pay for the full bill of the doctor). We call an event where a fixed number of people are sick, e.g., three out of ten, a *statistical state*. If, on the other hand, the insurance company has no idea of the risk, it faces 11 different risks according to the 11 different statistical states that may arise. So the company could offer insurance contracts that pay 11 different payoffs in case of sickness, contingent on the 11 different statistical states.[1] With this example, we see that all agents face two kinds of risks; first of all, the *individual risk* of becoming sick or not, and secondly the *collective risk*, faced by all agents together, of not knowing which statistical state will occur, i.e., how many people in the society will become sick.

Of course, this example, though explaining the difference between the two types of risks, is not quite realistic, since in the long run the collective risk can be derived from statistical data. More convincing examples of unknown risks are the effects of global warming or the depletion of the ozone layer on the income or health of individuals (see Chichilnisky and Heal [1993]).

As already mentioned, the crucial idea of Chichilnisky and Heal consists in combining two well-known financial instruments in order to insure those two different risks. First, *mutual insurance contracts* serve to insure the individual risk contingent on each possible statistical state of sick people among the population,[2] and second, *Arrow securities* serve to deal with the unknown collective risk, that is, the uncertainty about the statistical state. One Arrow security is needed for each possible number of sick people in the population.

In order to model the ignorance of the risk, Chichilnisky and Heal [1998] introduce consumers who differ not only in their preferences and in their initial endowments but also, above all, in their beliefs about the states of the world. In Chichilnisky and Heal [1992], the model is slightly more complicated and also allows for different types of consumers. In the present article, we follow the setup of Chichilnisky and Heal [1998]. But our results apply equally to the model in Chichilnisky and Heal [1992].

The first result of Chichilnisky and Heal [1998] roughly states that, if beliefs are the same for all households, complete insurance is possible within each statistical state. More precisely, in a competitive contingent equilibrium, the final bundle consumed by each agent depends on the statistical state only.

Secondly, however, the authors claim that if beliefs of any two households with different utility functions are different, complete insurance is impossible! For different beliefs, the authors claim that a weaker version of the result holds, i.e., complete insurance is only possible if the number of individual states does not exceed two *and* if the economy is regular.

In the present article, we argue that this second result of Chichilnisky and Heal is not correct, and we present a counterexample. More precisely, we construct a small economy in which consumers have *different* beliefs and nevertheless get fully insured in each statistical state. Moreover, we show that the full insurance result holds even if different consumers' beliefs about the states of the world are different. This finding makes the approach of Chichilnisky and Heal even more powerful, for the notion of *unknown risk* is associated in general with different opinions about certain risks, as is the case, for example, in the current discussion about global warming. Of course, unknown risk does not exclude the case that all agents have the same, possibly wrong beliefs. But this case is very special and unlikely. An approach on unknown risks should at least allow for people having different beliefs, for if complete insurance were not possible in general whenever—as is usual in politics—there are different opinions on certain hazardous risks such as global warming,[3] one could not guarantee that state-contingent market equilibria—known to be Pareto efficient—can be supported in the institutional framework of mutual insurance contracts and Arrow securities. In other words, we would not be able to conclude that an analogous result to the Second Welfare Theorem holds in this framework. And thus we could not derive the existence of Pareto-efficient market equilibria in such economies from the existence of contingent-market equilibria.[4] In order to make sure that Pareto-efficient allocations can be obtained in a decentralized way, we would be forced to hold onto the contingent market approach.

Our main result generalizes Chichilnisky and Heal's first result considerably by showing that identical beliefs are not necessary to guarantee full insurance. Even with different beliefs, households get fully insured within each statistical state in contingent-market equilibrium. By virtue of this result, we are able to generalize a further result of Chichilnisky and Heal by demonstrating that even with *different* beliefs, contingent-market equilibria can be achieved by mutual insurance contracts and Arrow securities. This result makes the combination of the two financial instruments, as suggested by Chichilnisky and Heal, more powerful.

The article is organized as follows. We set up the basic model in Section 2, and we present a generalized result on contingent equilibria in our framework in Section 3. In Section 4, we then introduce economies with securities and mutual insurance. Section 5 contains the generalization of Chichilnisky and Heal's second result about supporting contingent-market equilibria in economies with securities and mutual insurance contracts. In Section 6, we discuss the flawed result of Chichilnisky and Heal and present the counterexample. The last section concludes. The proofs are given in the Appendix.

2. Basics of the model

Consider a pure exchange economy with H consumers, and denote the set of consumers by $\tilde{H} := \{1, \ldots, H\}$. We introduce two types of states (of the world), namely, *individual states* and *collective states*. Let $\tilde{S} := \{1, \ldots, S\}$ be the set of *individual states*[5] that can occur for an individual consumer. Initial endowments depend on the individual state only, thus reflecting the individual component of the unknown individual risk. Moreover, it is assumed that this dependence is the same for all households. The *collective state* of the economy can be described by a list of the individual states of each consumer, i.e., an H-tuple $\omega = (s_1, \ldots, s_H)$, with $s_h \in \tilde{S}$ for all $h \in \tilde{H}$. Formally, a collective state ω is a function $\omega : \tilde{H} \to \tilde{S}$ with $h \mapsto \omega(h)$. Therefore, $\Omega := \{\omega \mid \omega : \tilde{H} \to \tilde{S}\}$ is the set of collective states. Clearly, $|\Omega| = S^H$.

By definition, $\omega(h)$ is the individual state of consumer h in the collective state ω, and $\tilde{H}_s^\omega := \{h \in \tilde{H} \mid \omega(h) = s\}$ is the set of all consumers who find themselves in the individual state s when the collective state is ω. Then the sets $(\tilde{H}_s^\omega)_{s\in\tilde{S}}$ form a partition of $\tilde{H}$, i.e.,

$$\dot{\bigcup_{s\in\tilde{S}}} \tilde{H}_s^\omega = \tilde{H} \quad \forall \omega \in \Omega. \tag{1}$$

Now define, for $s \in \tilde{S}$,

$$r_s(\omega) := \frac{|\tilde{H}_s^\omega|}{H} \tag{2}$$

as the proportion of all consumers who are in the individual state s when the collective state is ω. Clearly, $\sum_{s\in\tilde{S}} r_s(\omega) = 1$ by (1) and (2) so that $r(\omega) := (r_1(\omega), \ldots, r_S(\omega)) \in \Delta^S$ is the distribution of consumers over the individual states in the collective state ω. We call $r(\omega)$ a *statistical state*, since it contains the statistical information about how many consumers are in a certain individual state but does not specify *who* is in which individual state. Let $R := r(\Omega) \subset \Delta^S$ be the set of statistical states. By use of simple combinatorics, it is easy to prove that $|R| = \binom{H+S-1}{S-1}$. Clearly, $|R| < |\Omega|$ for $S \geq 2$.[6]

Given a certain statistical state, there is only individual risk. The unknownness of the individual risk is reflected in the model by consumers having different beliefs about the statistical state. Summing up, the unknown individual risk is represented by *collective uncertainty* about the statistical state and *individual risk* for a given statistical state.

We now turn to the subjective beliefs about the states of the world. For convenience, we start with subjective probability distributions over collective states, from which we will later derive the distributions over statistical states. Let Π^h denote the subjective probability distribution of consumer h over the collective states. In order to avoid problems caused by zero probabilities, we assume[7] that $\Pi_\omega^h := \Pi^h(\omega) > 0$ for all $h \in \tilde{H}$ and $\omega \in \Omega$.

The following assumption about the probabilities Π^h, originating in Malinvaud [1973] and being recalled by Chichilnisky and Heal, is important for the main results.

Assumption 1 (Anonymity Assumption): *For each consumer $h \in \tilde{H}$, we have*

$$\Pi_\omega^h = \Pi_{\hat{\omega}}^h \qquad \forall \omega, \hat{\omega} \in \Omega \quad \textit{with } r(\omega) = r(\hat{\omega}). \tag{3}$$

The Anonymity Assumption means that any two collective states leading to the same statistical state are considered equally likely by all consumers. If we define $\Omega_r := \{\omega \in \Omega \mid r(\omega) = r\}$ for $r \in R$, then the assumption implies—with a little abuse of notation[8]—that the subjective probabilities of each consumer are the same for all states in Ω_r, i.e.,

$$\Pi^h_\omega = \Pi^h_{\hat{\omega}} =: \Pi^h_r \quad \forall \omega, \hat{\omega} \in \Omega_r, \forall h \in \tilde{H}. \tag{4}$$

Every Π^h induces a corresponding probability distribution $\hat{\Pi}^h$ on R ($h \in \tilde{H}$) defined by $\hat{\Pi}^h_r := \sum_{\omega \in \Omega_r} \Pi^h_\omega$ ($\forall r \in R$). Note that $\dot{\bigcup}_{r \in R} \Omega_r = \Omega$ by definition of Ω_r.[9]

Since $\Pi^h > 0$ in each component, it follows that $\hat{\Pi}^h_r > 0$ ($\forall r \in R, \forall h \in \tilde{H}$). Using the Anonymity Assumption (here implicit in (4)), we obtain

$$\hat{\Pi}^h_r \overset{Def.}{=} \sum_{\omega \in \Omega_r} \Pi^h_\omega \overset{(4)}{=} \sum_{\omega \in \Omega_r} \Pi^h_r = |\Omega_r| \cdot \Pi^h_r. \tag{5}$$

Concerning the conditional probability $\Pi^h_{s|r}$ of consumer h for being in the individual state s given the statistical state r, we have the following:

Lemma 1:

$$\Pi^h_{s|r} = r_s \quad \forall h, s, r. \tag{6}$$

In words: the conditional probability $\Pi^h_{s|r}$ of consumer h for being in the individual state s given the statistical state r equals the proportion of consumers in the individual state s (given the statistical state r). See the Appendix for details of the proof, which uses the Anonymity Assumption.

Let $L \in I\!N$ be the number of commodities. Since consumers face the unknown individual risk, we are interested in consumption across collective states. Therefore, the commodity space formally becomes $I\!R^{LS^H}$, and the consumption set of consumer h is $I\!R^{LS^H}_+$. Let $z_h = (z_{h\omega})_{\omega \in \Omega} \in I\!R^{LS^H}_+$ denote a consumption vector of consumer h, and let $z_{h\omega} \in I\!R^L_+$ the consumption of h in the collective state ω.

Further, let $e^h_s = e_s \in I\!R^L_{++}$ be the initial endowment of consumer h when she or he is in the individual state s ($h \in \tilde{H}, s \in \tilde{S}$). Then the initial endowment e^h_ω of consumer h in the collective state ω is given by $e^h_{\omega(h)} \in I\!R^L_{++}$ ($h \in \tilde{H}, \omega \in \Omega$). Accordingly, $e^h := (e^h_{\omega(h)})_{\omega \in \Omega} \in I\!R^{LS^H}_{++}$ defines the initial endowment of h over all collective states.

About the preferences we make the following assumption:

Assumption 2: *The preferences of consumer h can be represented by a utility function* $W^h : I\!R^{LS^H}_+ \to I\!R$, *with*

$$W^h(z_h) := \sum_{\omega \in \Omega} \Pi^h_\omega \, U^h(z_{h\omega}), \tag{7}$$

where $U^h : I\!R^L_+ \to I\!R$ *is twice continuously differentiable, strictly increasing, and strictly concave for all* $h \in \tilde{H}$.[10]

Thus, U^h is a utility function of the von Neumann–Morgenstern type, whereas W^h denotes the expected utility function of consumer h. Assumption 2 implies that the consumer demand correspondences are continuous functions.[11]

For the sake of consumers' insurance against the unknown individual risk, we will consider two different kinds of frameworks: contingent markets, on the one hand, and a combination of securities markets and mutual insurance contracts on the other. Whereas the contingent-market approach is standard, the other approach—to the best of our knowledge—was introduced for the first time by Cass, Chichilnisky, and Wu [1996] and by Chichilnisky and Heal [1992, 1998]. As we will show later, this approach "fits the problem" and is clearly more suitable than the contingent-market approach. For a related approach to the allocation of collective and individual risks, see Magill and Shafer [1992].

3. Properties of contingent-market equilibrium

In this section, we present a result on the properties of contingent-market equilibrium given the assumptions made in our model. Although we do not consider contingent markets to be the appropriate framework for unknown risks, we use the concept as a benchmark in order to exploit its well-known properties for our main result. There we will show that a contingent-market equilibrium allocation can also be supported as an equilibrium in the corresponding economy with securities and mutual insurance.

Consider a complete system of contingent markets with regard to the set of collective states Ω. Since each collective state consists of a *list* of all individual states, there are many collective states—and the use of contingent markets *inflates* the commodity space to the dimension LS^H. Assuming the existence of a complete system of contingent markets with respect to the state space Ω is therefore tantamount to considering an Arrow–Debreu economy with LS^H commodities. The well-known results concerning the existence of equilibria and their properties, e.g., the Welfare Theorems, continue to hold. By the First Welfare Theorem, we know that contingent-market equilibria are Pareto efficient. On the other hand, the disadvantages of the contingent-market approach are obvious. Since the number of markets rises exponentially with the number of consumers, a complete system of contingent markets is unlikely to be established. And even if such a system were established, the large number of markets alone makes contingent market equilibrium a tedious concept to be dealing with.

Fortunately, given our assumptions, things are not as complicated as they seem: there is much redundancy in contingent-market equilibria, in the sense that equilibrium prices and allocations are constant for each statistical state, i.e., within certain groups of collective states. So in our context, the structure of contingent equilibria is actually simpler than could be expected in general. This outcome is formally stated in the following theorem, which generalizes Proposition 1 of Chichilnisky and Heal [1998].

Theorem 1: *Let* $(p^*, z^*) = ((p^*_\omega)_{\omega\in\Omega}, (z^*_{h\omega})_{\omega\in\Omega, h\in\tilde{H}}) \in I\!R^{LS^H}_{++} \times I\!R^{LS^H H}_{+}$ *be a contingent-market equilibrium. Then*

1. $z^*_\omega = z^*_r \quad \forall \omega \in \Omega_r,\ \forall r \in R,$ (8)

2. $p^*_\omega = p^*_r \quad \forall \omega \in \Omega_r, \ \forall r \in R,$ (9)

where $z^*_\omega := (z^*_{h\omega})_{h \in \tilde{H}} \in I\!R^{LH}_+$ *and* $z^*_h := (z^*_{h\omega})_{\omega \in \Omega} \in I\!R^{LS^H}_+$.

The proof is lengthy and is given in the Appendix. In contrast to Chichilnisky and Heal, however, we do not require all consumers to have the same probability distribution over collective states, i.e., we do *not* assume $\Pi^h = \Pi^k$ for $h, k \in \tilde{H}$. This extends the range of application of the model considerably, because if risks are *unknown*, it is extremely unlikely that all consumers have the same, possibly wrong probability beliefs. Thus, we also do not need to restrict the range of application to the special case of $S = 2$ and a regular economy, as Chichilnisky and Heal do when not assuming identical beliefs.

4. Economies with securities and mutual insurance contracts

As explained in Section 2, the unknown individual risk has a collective and an individual component. Whereas securities are suitable for collective uncertainty, the adequate way of dealing with individual risk is the use of insurance markets. For this reason, Chichilnisky and Heal introduce the appropriate asset for each component of the unknown individual risk, i.e., a complete set of *securities* defined on statistical states and a suitable set of *mutual insurance contracts* contingent on each statistical state. The securities enable consumers to insure themselves against the collective uncertainty with respect to the statistical state, whereas the remaining individual risk can be insured by mutual insurance contracts contingent on the statistical state.

Consider now a system of $|R|$ Arrow securities $A_r \in I\!R^{|R|}$ $(r \in R)$. Define A_r as the rth unit vector in $I\!R^{|R|} (r \in R)$. A_r is a contract guaranteeing one unit of the numéraire—good if the statistical state turns out to be r, and nothing otherwise.[12] The price of security A_r $(r \in R)$ is denoted by $q_r \in I\!R_+$. Let $q := (q_1, \ldots, q_{|R|}) \in I\!R^{|R|}_+$. Moreover, denote by $a^h_r \in I\!R$ the amount of security A_r bought (or sold in the case of $a^h_r < 0$) by consumer h. For $h \in \tilde{H}$, define $a^h := (a^h_1, \ldots, a^h_{|R|}) \in I\!R^{|R|}$ and $a := (a^1, \ldots, a^H) \in I\!R^{|R|H}$.

For each statistical state, consumer h chooses a vector of insurance transfers $m^h_r := (m^h_{1r}, \ldots, m^h_{Sr}) \in I\!R^S$, where $m^h_{sr} \in I\!R$ is the transfer received (or paid) by consumer h if the statistical state is r and his individual state is s. Each consumer chooses a tuple $m^h := (m^h_1, \ldots, m^h_{|R|})$ of insurance vectors.

Assumption 3: *Consumer h can choose his tuple of insurance vectors from the set*

$$M^h := \left\{ m^h \in I\!R^{S|R|} \;\middle|\; \sum_{s=1}^{S} \Pi^h_{s|r} m^h_{sr} = 0 \quad \forall r \in R \right\}.$$

In words: given a statistical state r, consumer h can choose any insurance vector m^h_r with contingent expectation of zero. This means the insurance vectors have to be *actuarially fair*.

The budget set of consumer h in an economy with securities and mutual insurance contracts (denoted by E_{SI}) at spot market prices p and security prices q is given by

$$B^h_{SI}(p,q;e^h) := \left\{ (z_h, a^h, m^h) \in \mathbb{R}^{LS^H}_+ \times \mathbb{R}^{|R|} \times \mathbb{R}^{S|R|} \;\middle|\; \begin{array}{l}(z_h, a^h, m^h) \text{ satis-}\\ \text{fies (10), (11), (12)}\end{array} \right\},$$

where

$$p_\omega\big(z_{h\omega} - e^h_\omega\big) = a^h_{r(\omega)} + m^h_{\omega(h)\ r(\omega)} \quad \forall \omega \in \Omega, \tag{10}$$

$$\sum_r q_r a^h_r = 0, \tag{11}$$

$$\sum_s \Pi^h_{s|r} m^h_{sr} = 0 \quad \forall r \in R. \tag{12}$$

Equation (10) represents the budget constraints of consumer h on the spot markets in the collective states, where the right-hand side consists of income (or obligations) from the portfolio of securities and the mutual insurance vector. Equation (11) ensures that the portfolio of consumer h is self-financing (there are no endowments in securities), whereas (12) requires the insurance vectors to be actuarially fair.

Sometimes we will only be interested in the first component of the tuples contained in the budget set. Hence we also consider the projection onto this component

$$\hat{B}^h_{SI}(p,q;e^h) := \left\{ z_h \in \mathbb{R}^{LS^H}_+ \;\middle|\; \begin{array}{l}\exists a^h \in \mathbb{R}^{|R|}, m^h \in \mathbb{R}^{S|R|}:\\ (z_h, a^h, m^h) \in B^h_{SI}(p,q;e^h)\end{array} \right\}.$$

Finally, we formally define the notion of equilibrium in an economy with securities and insurance. Doing this, we follow the intuition presented in the articles by Chichilnisky and Heal [1992, 1998] where no formal definition was given.

Definition 1: *An equilibrium in an economy with securities and mutual insurance contracts (equilibrium in E_{SI}) is a tuple $(p^*, q^*, z^*, a^*, m^*) \in \mathbb{R}^{LS^H}_{++} \times \mathbb{R}^{|R|}_{++} \times \mathbb{R}^{LS^H H}_+ \times \mathbb{R}^{|R|H} \times \mathbb{R}^{|R|SH}$ with*

(a) $\forall h = 1, \ldots, H$:

$$z^*_h \in \textit{argmax}\big\{ W^h(z^h) \,\big|\, z_h \in \hat{B}^h_{SI}(p^*, q^*; e^h) \big\} \tag{13}$$

$$\textit{and} \quad (z^*_h, a^{h*}, m^{h*}) \in B^h_{SI}(p^*, q^*; e^h), \tag{14}$$

(b) $$\sum_{h=1}^H (z^*_h - e^h) = 0, \tag{15}$$

(c) $$\sum_{h=1}^H a^{h\,*}_r = 0 \quad \forall r \in R. \tag{16}$$

Here, (13) and (14) require utility maximization with respect to the budget set $B^h_{SI}(p^*, q^*; e^h)$ at equilibrium prices. Equation (15) guarantees market clearing on the spot markets, whereas (16) ensures market clearing on the security markets.

Note that (12) and (14) guarantee that in each statistical state the sum of premia and payments are balanced, i.e.,

$$\sum_{h,s} \Pi^h_{s|r} m^{h*}_{sr} = 0 \quad \forall r \in R. \tag{17}$$

So this need not be required in the definition of equilibrium.

5. Supporting contingent-market equilibria in economies with securities and mutual insurance

We are now able to show that each contingent-market equilibrium can be supported as an equilibrium in the corresponding economy with securities and mutual insurance. From this result—since contingent-market equilibria are just Arrow–Debreu equilibria with an appropriately defined commodity space—we can learn a lot about equilibria in economies with securities and mutual insurance. Technically, since the E_{SI} equilibrium is defined on a larger space, we have to "extend" the contingent-market equilibrium by defining both security equilibrium prices as well as equilibrium portfolios plus insurance vectors for consumers such that we get an equilibrium in E_{SI}. In contrast to Chichilnisky and Heal [1998], we are again able to dispense with the assumption of equal beliefs across consumers—without having to assume $S = 2$ and a regular economy instead—thus generalizing their main result considerably.

If (p^*, z^*) is a contingent market equilibrium, then, using Theorem 1, Equation (8), we can write $z^{h*}_r := z^*_{h\omega}$ for $r := r(\omega)$. This notation reflects the result of Theorem 1. In other words, in a contingent-market equilibrium, consumption only depends on the statistical state.

Theorem 2: *Let $(p^*, z^*) \in \mathbb{R}^{LS^H}_{++} \times \mathbb{R}^{LS^H H}_{+}$ be a contingent-market equilibrium. Define $q^* \in \mathbb{R}^{|R|}_{++}$, $a^* \in \mathbb{R}^{|R|H}$, and $m^* \in \mathbb{R}^{S|R|H}$ by*

$$q^*_r := |\Omega_r| \quad \forall r \in R, \tag{18}$$

$$a^{h*}_r := \sum_{s=1}^{S} \Pi^h_{s|r} p^*_r \big(z^{h*}_r - e_s\big) \quad \forall r \in R, \forall h \in \tilde{H}, \tag{19}$$

$$m^{h*}_{sr} := p^*_r \big(z^{h*}_r - e_s\big) - a^{h*}_r \quad \forall r \in R, \forall s \in \tilde{S}, \forall h \in \tilde{H}. \tag{20}$$

Then $(p^*, z^*, q^*, a^*, m^*)$ is an equilibrium in the corresponding economy with securities and mutual insurance contracts (i.e., in E_{SI}).

The theorem is constructive in the sense that it not only proves that contingent-market equilibria can be supported in economies with securities and mutual insurance but also states explicitely *how* portfolios and insurance vectors as well as security prices have to be chosen. The equilibrium allocation of securities and insurance can be interpreted as follows (see Chichilnisky and Heal [1992, 1998]). a_r^{h*} is the expected value of the excess demand of consumer h, conditional on being in the statistical state r, so that on average his budget is balanced in the state r. Of course, the actual value of the excess demand can differ from the expected value. The mutual insurance contracts are designed to make up for these differences.[13]

Theorem 2 allows several important conclusions. First of all, it tells us that it is possible to have Pareto-efficient equilibria in E_{SI}. More precisely, if we have an E_{SI} equilibrium induced by a contingent equilibrium in the sense of Theorem 2, then the allocation of goods in this equilibrium is Pareto efficient. Considering this theorem in connection with the Second Welfare Theorem, we can conclude that every Pareto-efficient allocation can be supported (if necessary after a suitable redistribution of endowments) in E_{SI}. So the analogue to the Second Welfare Theorem holds in economies with securities and mutual insurance, whereas this is not true for the First Welfare Theorem. (From Theorem 2 we can only conclude that a certain kind of equilibrium in E_{SI} is Pareto efficient.)

Theorem 2 also implies that there are always at least as many equilibria in E_{SI} as in the underlying contingent economy. In particular, the existence of contingent equilibrium implies the existence of equilibrium in the corresponding E_{SI} economy. On the other hand, if the contingent-market equilibrium is not unique, the E_{SI} equilibrium cannot be unique either.

6. Discussion of Chichilnisky and Heal [1998]

In this section, we briefly summarize the results of Chichilnisky and Heal [1998] in order to be able to compare those results with ours. As already mentioned, Chichilnisky and Heal present a result corresponding to Theorem 1 (Proposition 1, p. 282), but with the additional assumption of beliefs being equal for all consumers. As discussed above, this assumption is not satisfactory when dealing with unknown risks.

Chichilnisky and Heal, however, even go one step further by claiming explicitly that the *full insurance within statistical states* result (c.f. our Theorem 1) does *not* hold for heterogenous beliefs, unless there are only two different individual states of the world and the economy is regular.

Chichilnisky/Heal [1998]: Proposition 2

(a) *If* $\Pi^h \neq \Pi^k$ *for some households* h, k *with* $U^h \neq U^k$, *then* (8) *does not hold.*

(b) *If the economy is regular, all agents have the same utilities, and* $S = 2$, *then one of the equilibrium prices must satisfy* $p^*_{\omega_1} = p^*_{\omega_2}$ *for all* ω_1, ω_2 *with* $r(\omega_1) = r(\omega_2)$.

Their main result (Theorem 1 of Chichilnisky and Heal [1998]) is a less general version of our Theorem 2 above—again, they either assume identical beliefs or only consider the special case of a regular economy with $S = 2$.

Part (a) of their Proposition 2 is essentially a negation of (8) in our Theorem 1. Since we have proved Theorem 1 for the more general case of heterogenous beliefs, our result contradicts the claim by Chichilnisky and Heal. In the following, we will explain what is wrong with their proof, and we will also present a counterexample to part (a) of their Proposition 2. Part (b) is correct, but has already turned out to be a special case of our Theorem 1. Therefore, we concentrate on part (a).

The line of argument in the proof (given on p. 288 of Chichilnisky and Heal [1998]) goes as follows.[14] The authors proceed indirectly by assuming (8) to hold and trying to derive a contradiction. They consider two households h and k, with $\Pi^h \neq \Pi^k$, and compute their marginal rates of substitution between consumption in two collective states ω_1 and ω_2, with $r(\omega_1) = r(\omega_2)$, i.e., two collective states leading to the same statistical state. Since Chichilnisky and Heal make an additional assumption on the separability of the utility function with respect to statistical states, they can derive household h's marginal rate of substitution between consumption in state ω_1 and ω_2 as $\Pi^h_{s_1|r}/\Pi^h_{s_2|r}$, where $s_1 = \omega_1(h)$ and $s_2 = \omega_2(h)$. By the same argument, household k's marginal rate of substitution is $\Pi^k_{s_1|r}/\Pi^k_{s_2|r}$. The contradiction, they say, then arises from the fact that these two households with allegedly different marginal rates of substitution face the same price vector.

Unfortunately, the proof contains two flaws. First of all, as Chichilnisky and Heal correctly say in their first paragraph on page 281,

$$\text{“[the] Anonymity [Assumption] implies that} \quad \Pi^h_{s|r} = r_s, \ldots\text{”.}^{15} \tag{21}$$

Therefore, in their construction of the proof, the marginal rates of substitution would be independent of the household considered and equal to r_{s_1}/r_{s_2} for *both* households h and k. So the contradiction vanishes, merely by making use of the implication (21). The second incorrectness is that the authors neglect the fact that in general, $s_1 = \omega_1(h) \neq \omega_1(k)$. Although this problem seems to be only a technical problem that could be corrected by defining $\omega_1(k) =: s_3$ and $\omega_2(k) =: s_4$, we would get

$$\frac{r_{s_1}}{r_{s_2}} = \frac{r_{s_3}}{r_{s_4}},$$

which does not give the required contradiction either.

Why does this line of argument not work in general? Consider the utility function as defined in (7) of the present article and on page 281 of Chichilnisky and Heal [1988].[16] In that notation, the marginal rate of substitution for household h between collective states ω_1 and ω_2 with $r(\omega_1) = r(\omega_2)$, when fully insured, is

$$\frac{\Pi^h_{\omega_1}}{\Pi^h_{\omega_2}} \overset{(3)}{=} 1$$

by the Anonymity Assumption. In the same way, we get for household k

$$\frac{\Pi^k_{\omega_1}}{\Pi^k_{\omega_2}} = 1.$$

This outcome, however, implies

$$MRS^h_{\omega_1\omega_2} = \frac{\Pi^h_{\omega_1}}{\Pi^h_{\omega_2}} = 1 = \frac{\Pi^k_{\omega_1}}{\Pi^k_{\omega_2}} = MRS^k_{\omega_1\omega_2}.$$

So clearly, since households' marginal rates of substitution are the same anyway, there can be no contradiction to the fact that households face the same price vector. This shows how powerful the Anonymity Assumption is.

Counterexample to part (a) of Proposition 2 of Chichilnisky/Heal [1998]

To make things even clearer, we present a counterexample to Proposition 2, part (a), of Chichilnisky and Heal [1998]. Take $H = 2$ and $S = 2$, so that the set of collective states is $\Omega = \{\omega_1, \dots, \omega_4\}$, where

$$\omega_1 = (1, 1), \quad \omega_2 = (1, 2), \quad \omega_3 = (2, 1), \quad \omega_4 = (2, 2).$$

There are three statistical states:

$$\begin{aligned} r(\omega_1) &= (1, 0) \ =: \ r_1, \\ r(\omega_2) = r(\omega_3) &= \left(\frac{1}{2}, \frac{1}{2}\right) \ =: \ r_2, \\ r(\omega_4) &= (0, 1) \ =: \ r_3. \end{aligned}$$

Set $e_1 = (1, 3)$ and $e_2 = (3, 1)$. The von Neumann–Morgenstern utility functions are given by

$$\begin{aligned} U^1(x_1, x_2) &= 1/4 \ln x_1 + 3/4 \ln x_2, \\ U^2(x_1, x_2) &= 3/4 \ln x_1 + 1/4 \ln x_2. \end{aligned}$$

The functions U^h are twice continuously differentiable, strictly increasing, and strictly concave on their domain $\mathbb{R}^2_{++}$. They satisfy Assumption 2 with the sole exception of only being defined on $\mathbb{R}^2_{++}$, since the natural logarithm is only defined for strictly positive values.[17] Note that, apart from the restriction of the domain, the functions U^h satisfy all the assumptions made by Chichilnisky and Heal [1998]. In particular the indifference curves in this example do not cut into the boundary. Let $\Pi^1 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and $\Pi^2 = (\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6})$. This leads to

$$\begin{aligned} p^* &= \left(\left(\frac{27}{164}, \frac{11}{164}\right), \left(\frac{45}{328}, \frac{39}{328}\right), \left(\frac{45}{328}, \frac{39}{328}\right), \left(\frac{9}{164}, \frac{33}{164}\right)\right), \\ z_1^* &= \left(\left(\frac{2}{3}, \frac{54}{11}\right), \left(\frac{4}{5}, \frac{36}{13}\right), \left(\frac{4}{5}, \frac{36}{13}\right), \left(2, \frac{18}{11}\right)\right), \quad \text{and} \\ z_2^* &= \left(\left(\frac{4}{3}, \frac{12}{11}\right), \left(\frac{16}{5}, \frac{16}{13}\right), \left(\frac{16}{5}, \frac{16}{13}\right), \left(4, \frac{4}{11}\right)\right). \end{aligned}$$

Note that the second and third components of the equilibrium price and the equilibrium allocation respectively (which correspond to the collective states ω_2 and ω_3, both leading to the same statistical state r_2) are identical. Therefore, only the statistical state matters for equilibrium—even with different beliefs. This conclusion contradicts part (a) of Proposition 2 in Chichilnisky and Heal.

7. Conclusions

We built on a recent contribution by Chichilnisky and Heal [1998], who considered exchange economies with unknown individual risks. These authors pointed out that ignorance of individual risks leads to additional collective risks. By introducing two financial instruments to deal with those two risks, i.e., mutual insurance contracts and Arrow securities, the authors showed that a (Pareto-efficient) state contingent Arrow–Debreu market equilibrium can be supported by an equilibrium in an economy endowed with those two financial instruments. However, they made the strong assumption of identical beliefs across households. Moreover, they claimed that if beliefs were different, complete insurance within a statistical state would be impossible, and only in a special case did they show that state contingent-market equilibria can be supported by the financial instruments discussed throughout their article. But this conclusion would leave us with only the contingent-market approach (with all its disadvantages) in order to restore efficiency.

In this article, we showed it not to be correct that different opinions on the risks lead to incomplete insurance within a statistical state. Even more strongly, we showed that a state-contingent Arrow–Debreu market equilibrium can still be supported in economies endowed with financial instruments such as *mutual insurance contracts* plus *Arrow securities*. Put differently, we showed that the analogue to the Second Welfare Theorem holds in such economies—even with different opinions on the risks. This finding makes the combination of those two instruments even more powerful.

It should be mentioned that this whole approach relies on von Neumann–Morgenstern utility functions. In particular, the households' utilities are state dependent only through different initial endowments, beliefs, and, thus, final allocations. The utility functions themselves are *not* state dependent. This situation may yet be considered as a shortcoming of the approach. For example, the utility of some medicine or of a wheelchair for a sick or handicapped person is in general different from the utility of these items for a "healthy" person. Thus, in a more general framework, we would like to deal with utility functions that differ from state to state.[18] With state-dependent utility functions, neither our proofs nor those of Chichilnisky and Heal go through. Whether the results or modified versions will hold for state-dependent utility functions is an open problem and has to be left for further research.

Appendix

Proof of Lemma 1

Step 1: Making use of elementary methods from combinatorics, it can be shown that

$$|\Omega_r| = \frac{H!}{(H \cdot r_1)! \cdot \ldots \cdot (H \cdot r_S)!} \quad \forall r \in R. \tag{22}$$

Step 2: For $h \in \tilde{H}, r \in R$ and $s \in \tilde{S}$, define $\Omega^h_{rs} := \{\omega \in \Omega_r \mid \omega(h) = s\}$ as the set of all collective states in Ω_r in which consumer h is in the individual state s. This defines a partition of Ω_r for each h and each r.

Arguing similarly as in step 1 proves that

$$\left|\Omega^h_{rs}\right| = r_s \cdot |\Omega_r| \quad \forall h \in \tilde{H},\ r \in R,\ \forall s \in \tilde{S}\,. \tag{23}$$

Step 3: The conditional probability $\Pi^h_{s|r}$ can now be calculated (in terms of collective states) as follows:

$$\begin{aligned}\Pi^h_{s|r} &= \Pi^h\{\omega \in \Omega : \omega(h) = s \mid r(\omega) = r\} = \frac{\Pi^h\{\omega \in \Omega : \omega(h) = s, r(\omega) = r\}}{\Pi^h\{\omega \in \Omega : r(\omega) = r\}}\\ &= \frac{\sum_{\omega\in\Omega^h_{rs}} \Pi^h_\omega}{\sum_{\omega\in\Omega_r} \Pi^h_\omega} \overset{(3)}{=} \frac{\sum_{\omega\in\Omega^h_{rs}} \Pi^h_r}{\sum_{\omega\in\Omega_r} \Pi^h_r} = \frac{\Pi^h_r}{\Pi^h_r} \cdot \frac{\left|\Omega^h_{rs}\right|}{|\Omega_r|} \overset{(23)}{=} \frac{r_s|\Omega_r|}{|\Omega_r|}\\ &= r_s,\end{aligned} \tag{24}$$

thus completing the proof.

Proof of Theorem 1:

Lemma: *For all $\omega \in \Omega_r$ and for all $r \in R$,*

$$\sum_{h=1}^{H} e^h_\omega = H \sum_{s=1}^{S} r_s \cdot e_s =: E_r. \tag{25}$$

That is, total endowments of the economy are the same for all collective states leading to the same statistical state.

The proof is easy and is omitted. It uses a number of suitable partitions as well as the definition of the function r_s.

Step 1: Assume that (8) does *not* hold; then

$$\exists h \in \tilde{H}, r \in R, \omega_1, \omega_2 \in \Omega_r \quad \text{with } z^*_{h\omega_1} \neq z^*_{h\omega_2}. \tag{26}$$

Starting from the equilibrium allocation z^*, we now define a different feasible allocation and show that it is a Pareto improvement on z^*. The First Welfare Theorem then gives us the desired contradiction.

Step 1a: For $h \in \tilde{H}$ and $r \in R$, we will define the conditional expectation Ez^*_{hr} of the $(z^*_{h\omega})_{\omega\in\Omega}$ given r. Let $\Pi^h(\omega\,|\,r)$ denote as usual the conditional probability of ω given r for consumer h. In a similar way, $\Pi^h(\omega, r)$ is the probability of the collective state being ω and the statistical state being r. By definition of Ω_r, we have

$$\Pi^h(\omega\,|\,r) = 0 \quad \forall \omega \notin \Omega_r, \tag{27}$$

and

$$\Pi^h(\omega, r) = \Pi^i_\omega \quad \forall \omega \in \Omega_r, \tag{28}$$

for all $h \in \tilde{H}$.

$$Ez^*_{hr} := \sum_{\omega\in\Omega} \Pi^h(\omega\,|\,r)\, z^*_{h\omega} \overset{(27)}{=} \sum_{\omega\in\Omega_r} \frac{\Pi^h(\omega, r)}{\hat{\Pi}^h(r)}\, z^*_{h\omega}$$
$$\overset{(28)}{=} \sum_{\omega\in\Omega_r} \frac{\Pi^h_\omega}{\hat{\Pi}^h_r}\, z^*_{h\omega} \overset{(4)}{=} \sum_{\omega\in\Omega_r} \frac{\Pi^h_r}{\hat{\Pi}^h_r}\, z^*_{h\omega} \overset{(5)}{=} \sum_{\omega\in\Omega_r} \frac{1}{|\Omega_r|}\, z^*_{h\omega}. \tag{29}$$

Define $Ez^*_h := (Ez^*_{hr(\omega)})_{\omega\in\Omega} \in I\!R^{LS^H}_+$ and $Ez^* := (Ez^*_1, \ldots, Ez^*_H)$. The bundle $Ez^*_{hr(\omega)}$ gives consumer h the conditional expectation of the bundles $(z^*_{h\omega})_{\omega\in\Omega}$—conditional on $r(\omega)$—instead of $z^*_{h\omega}$. Since Ez^*_{hr} does not depend on Π^h for all $h \in \tilde{H}$ and for all $r \in R$, Ez^* is also independent of the consumers' probability distributions.

Step 1b: We show that Ez^* is a feasible allocation. Let $r \in R$.

$$\sum_h Ez^*_{hr} \overset{(29)}{=} \sum_h \sum_{\omega\in\Omega_r} \frac{1}{|\Omega_r|}\, z^*_{h\omega} = \sum_{\omega\in\Omega_r} \frac{1}{|\Omega_r|} \sum_h z^*_{h\omega}. \tag{30}$$

By assumption, z^* is an equilibrium allocation. Hence, market clearing yields $\sum_h z^*_{h\omega} = \sum_h e^h_\omega$ for all $\omega \in \Omega$. By (25), we have $\sum_h e^h_\omega = E_r$ for all $\omega \in \Omega_r$. Hence, $\sum_h z^*_{h\omega} = \sum_h e^h_\omega = E_r$ for all $\omega \in \Omega_r$. Substituting into (30) yields (for each $\omega \in \Omega_r$)

$$\sum_h Ez^*_{hr} = \sum_{\omega\in\Omega_r} \frac{1}{|\Omega_r|} E_r = E_r \cdot \frac{1}{|\Omega_r|} \sum_{\omega\in\Omega_r} 1 = E_r = \sum_h e^h_\omega\,. \tag{31}$$

Hence,

$$\sum_h Ez^*_h = \left(\sum_h Ez^*_{hr(\omega_1)}, \ldots, \sum_h Ez^*_{hr(\omega_{S^H})} \right)$$
$$= \left(\sum_h e^h_{\omega_1}, \ldots, \sum_h e^h_{\omega_{S^H}} \right) = \sum_h e^h\,.$$

This completes the proof of Ez^* being a feasible allocation.

Step 1c: We now show that Ez^* is a Pareto improvement on z^*.

1. It is easily shown that for $h = 1, \ldots, H$,

$$\left[z^*_{h\omega} = z^*_{hr} \quad \forall\omega \in \Omega_r \quad \forall r\right] \Longleftrightarrow Ez^*_h = z^*_h. \tag{32}$$

2. We now consider the following statement:

$$\left[h \text{ satisfies: } z^*_{h\omega_1} = z^*_{h\omega_2} = z^*_{hr} \quad \forall\omega_1, \omega_2 \in \Omega_r, \forall r\right]. \tag{33}$$

Let $h \in \tilde{H}$. Then two cases are possible:

Case 1: h satisfies (33), which by point 1 is equivalent to $Ez_h^* = z_h^*$. But then

$$W^h(Ez_h^*) = W^h(z_h^*).$$

Case 2: h does *not* satisfy (33), i.e.,

$$\exists r \in R \text{ and } \omega_1 \text{ and } \omega_2 \in \Omega_r, \quad \text{with } z^*_{h\omega_1} \neq z^*_{h\omega_2}. \tag{34}$$

To start with,

$$\begin{aligned} W^h(Ez_h^*) &\overset{(4),(7)}{=} \sum_r \Pi_r^h \sum_{\omega\in\Omega_r} U^h(Ez^*_{hr}) \\ &\overset{(29)}{=} \sum_r \Pi_r^h \sum_{\omega\in\Omega_r} U^h\left(\sum_{\omega\in\Omega_r} \frac{1}{|\Omega_r|}\, z^*_{h\omega}\right). \end{aligned} \tag{35}$$

Since $\sum_{\omega\in\Omega_r} \frac{1}{|\Omega_r|} = 1$, the argument of U^h in (35) is a convex combination of the $z^*_{h\omega}$, $\omega \in \Omega_r$. In order to distinguish easily between nontrivial and trivial convex combinations, we define the sets

$$R^h_= := \{r \in R \mid z^*_{h\omega} = z^*_{hr} \quad \forall\omega \in \Omega_r\}$$

and

$$R^h_{\neq} := \left\{r \in R \mid \exists\omega_1, \omega_2 \in \Omega_r \text{ with } z^*_{h\omega_1} \neq z^*_{h\omega_2}\right\}.$$

Then $R^h_= \dot\cup R^h_{\neq} = R$. For all $r \in R^h_=$, we are dealing with a trivial convex combination, i.e., $\sum_{\omega\in\Omega_r} \frac{1}{|\Omega_r|} z^*_{h\omega} = z^*_{hr}$. Consequently, for all $r \in R^h_=$,

$$U^h\left(\sum_{\omega\in\Omega_r} \frac{1}{|\Omega_r|} z^*_{h\omega}\right) = U^h(z^*_{hr}) = \sum_{\omega\in\Omega_r} \frac{1}{|\Omega_r|} U^h(z^*_{h\omega}). \tag{36}$$

For $r \in R^h_{\neq}$, we have nontrivial convex combinations. By strict concavity of U^h,

$$U^h\left(\sum_{\omega\in\Omega_r} \frac{1}{|\Omega_r|} z^*_{h\omega}\right) > \sum_{\omega\in\Omega_r} \frac{1}{|\Omega_r|} U^h(z^*_{h\omega}) \quad \forall r \in R^h_{\neq}. \tag{37}$$

Since $R^h_{\neq} \neq \emptyset$ by (34),

$$W^h(Ez_h^*) \overset{(35)}{=} \sum_r \Pi_r^h \sum_{\omega\in\Omega_r} U^h\left(\sum_{\psi\in\Omega_r} \frac{1}{|\Omega_r|} z^*_{h\psi}\right)$$

$$
\begin{aligned}
&\overset{(36),(37)}{>} \sum_r \Pi_r^h \sum_{\omega\in\Omega_r} \underbrace{\sum_{\psi\in\Omega_r} \frac{1}{|\Omega_r|} U^h(z^*_{h\psi})}_{\textit{indep. of}\,\omega} \\
&= \sum_r \sum_{\psi\in\Omega_r} \frac{1}{|\Omega_r|} U^h(z^*_{h\psi}) \sum_{\omega\in\Omega_r} \Pi_r^h \\
&\overset{(4)}{=} \sum_{\omega\in\Omega} \Pi_\omega^h U^h(z^*_{h\omega}) = W^h(z_h^*).
\end{aligned}
$$

3. By point 2 we have $W^h(Ez_h^*) \geq W^h(z_h^*)$ for *all* $h \in \tilde{H}$. Due to (26), there exists at least one h that does not satisfy (33). For this/these h, we even have $W^h(Ez_h^*) > W^h(z_h^*)$. Therefore, Ez^* is a Pareto improvement on z^*.

Step 1d: By the First Welfare Theorem, z^* is Pareto efficient, and thus there cannot be an allocation that is a Pareto improvement on z^*. This gives us the desired contradiction, and (8) must hold.

Note that (8) also implies that within each statistical state, the equilibrium consumption of consumer h does not depend on the individual state, i.e., in contingent equilibrium we have full insurance within each statistical state. Since consumers are risk averse according to Assumption 2, this is not surprising.

Step 2: Since (p^*, z^*) is a contingent-market equilibrium, we know that consumers maximize utilities. Let $h \in \tilde{H}$. From the first-order conditions for utility maximization of consumer h, we get

$$
\Pi_\omega^h \cdot \text{grad}\, U^h(z^*_{h\omega}) = \lambda_h \; p^*_\omega \quad \forall \omega \in \Omega, \tag{38}
$$

where λ_h is the Lagrange multiplier in the maximization problem of consumer h. Using the Anonymity Assumption, Equation (38) yields

$$
p^*_{\omega_1} = \frac{\Pi_r^h}{\lambda_h} \cdot \text{grad}\, U^h\big(z^*_{h\omega_1}\big)
$$

and

$$
p^*_{\omega_2} = \frac{\Pi_r^h}{\lambda_h} \cdot \text{ grad}\, U^h\big(z^*_{h\omega_2}\big) \overset{(8)}{=} \frac{\Pi_r^h}{\lambda_h} \cdot \text{grad}\, U^h\big(z^*_{h\omega_1}\big) = p^*_{\omega_1}\,,
$$

which proves (9). This completes the proof.

Proof of Theorem 2:

According to Definition 1, we have to show that $(p^*, z^*, q^*, a^*, m^*)$ satisfies (13) to (16).

Step 1: We use the fact that (p^*, z^*) is a contingent-market equilibrium in order to derive some useful identities.

Step 1a: By Theorem 1,

$$p^*_\omega = p^*_r \quad \forall \omega \in \Omega_r, \forall r, \tag{39}$$

and

$$z^*_{h\omega} = z^{h*}_r \quad \forall \omega \in \Omega_r, \forall r. \tag{40}$$

Step 1b: By our assumptions, z^*_h satisfies the budget restraint of consumer h for all $h \in \tilde{H}$. Due to Theorem 1 (Equations (39) and (40)), as well as by the definition of Ω^h_{rs} and (23), we can express this equation in terms of r and s instead of ω:

$$\begin{aligned} p^*(z^*_h - e^h) &\overset{(39),(40)}{=} \sum_{r \in R} p^*_r \sum_{\omega \in \Omega_r} \left(z^{h*}_r - e^h_{\omega(h)}\right) \\ &\overset{(23)}{=} \sum_{r \in R} p^*_r \sum_{s=1}^{S} r_s |\Omega_r| \left(z^{h*}_r - e^h_s\right). \end{aligned}$$

Therefore,

$$p^*(z^*_h - e^h) = 0 \Longleftrightarrow \sum_{r \in R} p^*_r |\Omega_r| \sum_{s=1}^{S} \Pi^h_{s|r} \left(z^{h*}_r - e^h_s\right) = 0. \tag{41}$$

Step 1c: We deal with the market-clearing condition in the same way. Here, besides Theorem 1, we use (1) and (2), obtaining

$$\begin{aligned} &\sum_h (z^*_h - e^h) = 0 \\ \Longleftrightarrow &\sum_h \left(z^*_{h\omega} - e^h_\omega\right) = 0 \quad \forall \omega \in \Omega \\ \Longleftrightarrow &\sum_h z^*_{h\omega} - \sum_h e^h_\omega = 0 \quad \forall \omega \in \Omega_r, \forall r \\ \overset{(25),(40)}{\Longleftrightarrow} &\sum_h z^{h*}_r - H \sum_s r_s e_s = 0 \quad \forall r \\ \Longleftrightarrow &\sum_h z^{h*}_r \underbrace{\left(\sum_s r_s\right)}_{=1} - \sum_h \sum_s r_s e_s = 0 \quad \forall r \\ \Longleftrightarrow &\sum_{h,s} r_s (z^{h*}_r - e_s) = 0 \quad \forall r. \end{aligned} \tag{42}$$

Step 2: First of all, we show that (14) is satisfied for all $h \in \tilde{H}$, i.e., we show that $(z^*_h, a^{h*}, m^{h*}) \in B^h_{SI}(p^*, q^*; e^h)$ for all $h \in \tilde{H}$.

Let $h \in \tilde{H}$. By definition of $B^h_{SI}(p^*, q^*; e^h)$, we have to show that (z^*_h, a^{h*}, m^{h*}) satisfies (10), (11), and (12) with reference to p^* and q^*.
Step 2a: We begin with (10). It is straightforward to check that (10) is satisfied with respect to p^* by the definition of m^{h*}_{sr} in (20) and Theorem 1.
Step 2b: Now we show that (11) holds with respect to q^*:

$$\sum_r q^*_r a^{h*}_r \overset{(18),(19)}{=} \sum_r |\Omega_r| \sum_s p^*_r \, \Pi^h_{s|r}\big(z^{h*}_r - e^h_s\big) \overset{(41)}{=} 0$$

Step 2c: It remains to prove that (12) also holds with respect to p^* for (z^*_h, a^{h*}, m^{h*}). Let $r \in R$. Then

$$\begin{aligned}
\sum_{s=1}^{S} \Pi^h_{s|r} m^{h*}_{sr} &\overset{(19),(20)}{=} \sum_{s=1}^{S} \Pi^h_{s|r}\left[p^*_r\big(z^{h*}_r - e^h_s\big) - \sum_{\sigma=1}^{S} \Pi^h_{\sigma|r}\big(p^*_r\big(z^{h*}_r - e^h_\sigma\big)\big)\right] \\
&= \sum_{s=1}^{S} \Pi^h_{s|r}\big[p^*_r\big(z^{h*}_r - e^h_s\big)\big] - \sum_{s=1}^{S} \Pi^h_{s|r} \underbrace{\left[\sum_{\sigma=1}^{S} \Pi^h_{\sigma|r}\big(p^*_r\big(z^{h*}_r - e^h_\sigma\big)\big)\right]}_{independ.\ of\ s} \\
&= \sum_{s=1}^{S} \Pi^h_{s|r}\big[p^*_r\big(z^{h*}_r - e^h_s\big)\big] - \left[\sum_{\sigma=1}^{S} \Pi^h_{\sigma|r}\big(p^*_r\big(z^{h*}_r - e^h_\sigma\big)\big)\right] \underbrace{\sum_{s=1}^{S} \Pi^h_{s|r}}_{=1} \\
&= 0,
\end{aligned}$$

i.e., (12) holds with reference to p^*. Thus, $(z^*_h, a^{h*}, m^{h*}) \in B^h_{SI}(p^*, q^*; e^h)$ for all $h \in \tilde{H}$, i.e., (14) holds.
Step 3: Now we show (13), i.e.,

$$z^*_h \in \operatorname{argmax}\big\{W^h(z_h) \mid z_h \in \hat{B}^h_{SI}(p^*, q^*; e^h)\big\} \quad \forall h \in \tilde{H}.$$

Step 3a: We begin by proving

$$\hat{B}^h_{SI}(p^*, q^*; e^h) \subseteq \bar{B}^h(p^*, e^h) := \big\{z_h \in \mathbb{R}^{LS^H}_+ \mid p^*\big(z_h - e^h\big) = 0\big\} \quad \forall h \in \tilde{H}. \qquad (43)$$

$\bar{B}^h(p^*, e^h)$ is the budget set of consumer h at prices p^* in the corresponding contingent-market economy.

Let $h \in \tilde{H}$ and $z_h \in \hat{B}^h_{SI}(p^*, q^*; e^h)$. By definition of $\hat{B}^h_{SI}(p^*, q^*; e^h)$, there exist $a^h \in \mathbb{R}^{|R|}$ and $m^h \in \mathbb{R}^{S|R|}$ such that $(z_h, a^h, m^h) \in B^h_{SI}(p^*, q^*; e^h)$. Consequently, (z_h, a^h, m^h) satisfies (10) with reference to p^*. Summing over ω leads to

$$\sum_{\omega\in\Omega} p^*_\omega\big(z_{h\omega} - e^h_\omega\big) = \sum_{\omega\in\Omega} a^h_{r(\omega)} + \sum_{\omega\in\Omega} m^h_{\omega(h)r(\omega)}$$

$$\iff \quad p^*(z_h - e^h) = \underbrace{\sum_r \sum_{\omega\in\Omega_r} a^h_{r(\omega)}}_{=0 \; by \; (18),(11)} + \sum_r \sum_{\omega\in\Omega_r} m^h_{\omega(h)r(\omega)}$$

$$\iff \quad p^*(z_h - e^h) = \sum_{r\in R} \sum_s \sum_{\omega\in\Omega^h_{rs}} m^h_{sr} \overset{(23)}{=} \sum_{r\in R} |\Omega_r| \underbrace{\sum_s \Pi^h_{s|r} m^h_{sr}}_{=0 \; by \; (12)}$$

$$= 0. \tag{44}$$

So $z_h \in \bar{B}^h(p^*, e^h)$, which proves (43).
Step 3b: Let $h \in \tilde{H}$. By assumption, we know that

$$z^*_h = \text{argmax}\{W^h(z_h) \mid z_h \in \bar{B}^h(p^*, e^h)\}$$

holds.[19] Since $\hat{B}^h_{SI}(p^*, q^*; e^h) \subseteq \bar{B}^h(p^*, e^h)$ and $z^*_h \in \hat{B}^h_{SI}(p^*, q^*; e^h)$, since the tuple (z^*_h, a^{h*}, m^{h*}) belongs to $B^h_{SI}(p^*, q^*; e^h)$ by Step 2, we directly get $z^*_h = \text{argmax}\{W^h(z_h) \mid z_h \in \hat{B}^h_{SI}(p^*, q^*; e^h)\}$, which proves (13).[20]
Step 4: Regarding (15), there is nothing to show, since $\sum_h (z^*_h - e_h) = 0$ continues to hold by assumption. Only (16) remains to be shown. Let $r \in R$. Then

$$\sum_h a^{h*}_r \overset{(19)}{=} \sum_h \sum_{s=1}^{S} \Pi^h_{s|r} \, p^*_r \big(z^{h*}_r - e^h_s\big) \overset{(6)}{=} p^*_r \sum_h \sum_s r_s \big(z^{h*}_r - e_s\big) \overset{(42)}{=} 0.$$

This completes the proof.

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Notes

1. Insurance companies that insure houses against damage from hurricanes usually have clauses saying that they are not obliged to pay for the full damage if there has been an extraordinarily strong hurricane. Whether the hurricane has been extraordinarily strong is defined by the number of houses that have been damaged. Thus, those companies offer contracts contingent on statistical states.
2. Those contracts are usually offered by insurance companies who know the risk.
3. See the discussion of Chichilnisky and Heal [1992].
4. The assumptions made in this article and in Chichilnisky and Heal [1992, 1998] also guarantee the existence of a contingent-market equilibrium (see Aliprantis, Brown, and Burkinshaw [1990]).
5. Without loss of generality, this set is assumed to be the same for all consumers.
6. Note that $|R|$ is a *polynomial* in H, whereas $|\Omega|$ increases *exponentially* with H. This point will be of importance later on, when we need financial assets for each state.
7. This assumption is not explicitly mentioned by Chichilnisky and Heal but seems to be required for their proofs also.
8. There should be no confusion by using the same symbol for the function $r(\cdot)$ and a statistical state $r \in R$.

9. Note also that $\hat{\Pi}_r^h \neq \Pi_r^h$ whenever $|\Omega_r| > 1$.
10. Chichilnisky and Heal only assume strict quasi-concavity, but in order for their proof to go through, they really need strict concavity. They make the additional assumption that indifference surfaces do not cut into the boundary. This assumption, however, is only required if strict monotonicity or strict concavity do not hold on the boundary of the $I\!R_+^L$. Therefore, under the assumptions made above, it is superfluous.
11. See Aliprantis, Brown, and Burkinshaw [1990], Theorem 1.3.8 on page 24.
12. Restricting our attention to pure Arrow securities simplifies notation, but there is no loss of generality.
13. m_{sr}^{h*} is the difference between the actual value of the excess demand if the statistical state is r and the individual state s and the expected value of the excess demand for the state r.
14. In order to avoid confusion, we continue to use our notation.
15. The Anonymity Assumption is made on page 280 in the fourth paragraph of Chichilnisky and Heal [1998].
16. The additional separability assumption in the latter article is not really needed.
17. By a slight, though rather technical, modification of U^h in a neighborhood of the boundary, one could achieve that U^h exactly meets Assumption 2. Since the consumers' demand—all consumers having strictly positive endowments—would never be on the boundary of $I\!R_+^8$ anyway, this modification is not necessary.
18. Chichilnisky and Heal [1992] start to write down utility functions with subscripts indicating the individual state. Later, they drop the subscripts. In Chichilnisky and Heal [1998], the utility functions are *not* state dependent.
19. By Assumption 2 there is a unique maximizer.
20. In this case there is even a unique maximizer. See (13).

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