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Infrequent Extreme Risks

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Abstract

The main tools and concepts of financial and actuarial theory are designed to handle standard, or even small risks. The aim of this paper is to reconsider some selected financial problems, in a setup including infrequent extreme risks. We first consider investors maximizing the expected utility function of their future wealth, and we establish the necessary and sufficient conditions on the utility function to ensure the existence of a non degenerate demand for assets with extreme risks. This new class of utility functions, called LIRA, does not contain the classical HARA and CARA utility functions, which are not adequate in this framework. Then we discuss the corresponding asset supply-demand equilibrium model.

Key words: extreme risk, reinsurance, risk measure, LIRA utility function

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1. Introduction

The main tools and concepts of financial theory have been designed for standard, or even small risks. For instance the definitions of the absolute or relative risk aversion coefficients are derived from an expansion of the expected utility in a neighbourhood of the certainty hypothesis (Arrow [1965]); the mean-variance approach is also justified by local risk arguments (Samuelson [1970]; Levy and Markowitz [1979]; Pulley [1981]; Epstein [1985]; Kimball [1990]); the limiting analysis of the binomial tree (Cox, Ross and Rubinstein [1979]) and the derivation of the standard Black-Scholes formula (Black and Scholes [1973]) also assume the absence of extreme risks in a small time interval. The aim of this note is to reconsider some standard financial problems, in the presence of infrequent extreme risks. In a two state framework such a pure infrequent risky claim is defined by:

$$Y = \begin{cases} 1 + r + u, & \text{with probability } 1 - p, \\ 1 + r + \dfrac{d}{p}, & \text{with probability } p, \end{cases}$$

where r is the riskfree interest rate, u (up) is positive, d (down) is negative, and the probability p is tending to zero. The risky claim has an expectation given by: $EY = 1+r+(1-p)u+d \sim$

$1 + r + u + d$ for small p, and a variance $VY = (1-p)u^2 + \frac{d^2}{p} - ((1-p)u + d)^2$, which tends to infinity when p tends to zero.

In Section 2, we consider investors maximizing the expected utility of their future wealth, and building a portfolio based on a riskfree asset and a risky asset whose returns contain infrequent extreme values. We establish the necessary and sufficient conditions on the utility function ensuring the existence of a non degenerate individual demand (or supply) in the risky asset. This class of utility functions is studied in Section 3; we derive a representation theorem for these utility functions, provide an interpretation of the expected utility as an average of put option prices and discuss several examples of LIRA utility functions. In Section 4, we analyse the corresponding equilibrium model and discuss the equilibrium price distributions.

2. The individual demand or supply for infrequent extreme risks

The aim of this section is to derive a necessary and sufficient condition on the utility function ensuring a non degenerate demand (or supply) for infrequent extreme risks. This condition, which is not satisfied by standard utility functions, should be introduced for the analysis of the investors' behaviour or the equilibrium condition on markets with extreme risks, such as the junk bonds or the reinsurance markets. To simplify the derivation of the condition we focus on dichotomous infrequent extreme risks, and discuss at the end of the section the case of continuous risks.

2.1. The restrictions on the utility function

Let us consider a market with a riskfree asset and an asset featuring large risk with a small probability. At the initial investment date the prices are normalized to one. The payoffs at the next date are $1 + r$, where r is the riskfree rate, and[1]:

$$Y = \begin{cases} 1 + r + u, & \text{with probability } 1 - p, \\ 1 + r + \dfrac{d}{p}, & \text{with probability } p, \end{cases} \tag{2.1}$$

where $u > 0, d < 0$.

Let us now consider an individual investor with an initial budget w, and a utility function U, which is defined on the real line, increasing, concave and differentiable. The individual investor selects a portfolio satisfying:

$$\begin{cases} \max\limits_{\theta_o, \theta} & EU[\theta_o(1 + r) + \theta Y] \\ \text{s.t.} & \theta_o + \theta = w. \end{cases} \tag{2.2}$$

Taking into account the budget constraint, the maximisation can be performed with respect to the allocation in the risky asset only:

$$\max_{\theta} EU[w(1 + r) + \theta(Y - (1 + r))]. \tag{2.3}$$

The optimization becomes:

$$\max_{\theta}(1-p)U[w(1+r)+\theta u]+pU\left[w(1+r)+\theta\frac{d}{p}\right]. \tag{2.4}$$

The propositions below provide restrictions ensuring a finite solution of the optimization for a fixed p and for p tending to zero, respectively. These propositions involve the limiting values $U'(+\infty)$ and $U'(-\infty)$. These values are well defined. $U'(+\infty)$ is always finite and nonnegative, whereas $U'(-\infty)$ can be infinite.

Proposition 2.1: *The optimization in* (2.4) *admits a finite solution $\theta_p(w, u, d, r)$ for p sufficiently small, if and only if*:

$$\frac{U'(-\infty)}{U'(+\infty)} > -\frac{d}{u} > \frac{U'(+\infty)}{U'(-\infty)}.$$

Proof. The values of the derivative of the objective function for $\theta = \pm\infty$ are:

$$u(1-p)U'(-\infty) + dU'(+\infty), \quad \text{for } \theta = -\infty,$$
$$u(1-p)U'(+\infty) + dU'(-\infty), \quad \text{for } \theta = +\infty.$$

The objective function is concave with respect to θ. Therefore the problem admits a finite solution iff the derivatives at $-\infty$ and $+\infty$ are positive and negative, respectively. This provides the inequality restrictions in Proposition 2.1. □

Note that the previous conditions are satisfied for any levels u and d if $U'(+\infty) = 0$.

We are also interested in the limiting behaviour of the optimal risky allocation θ_p, when p tends to zero.

Proposition 2.2: *Under the condition of Proposition* 2.1, *a necessary condition for θ_p to have a strictly positive finite limit, when p tends to zero, is*: $U'(-\infty) = \lim_{W\to-\infty} U'(W) < +\infty$.

Proof. see the Appendix. □

The necessary condition involves only the utility function and neither the initial budget, nor the riskfree rate, nor the up and down values of the risky return.

2.2. *The individual demand (or supply)*

The individual demand (or supply) is given by the proposition below, proved in the Appendix.

Proposition 2.3: *Under the conditions of Propositions* 2.1–2.2, *θ_p always converges to a finite limit θ^*. Moreover we have*:

(i) $\theta^* < 0$, *that is a supply*, *if*:

$$\frac{U'(-\infty)}{U'(+\infty)} > -\frac{d}{u} > \frac{U'[w(1+r)]}{U'(+\infty)},$$

and then:

$$\theta^* = \frac{1}{u}\left\{U'^{-1}\left[-\frac{d}{u}U'(+\infty)\right] - w(1+r)\right\}.$$

(ii) $\theta^* = 0$, *that is a riskfree portfolio*, *if*:

$$\frac{U'[w(1+r)]}{U'(+\infty)} > -\frac{d}{u} > \frac{U'[w(1+r)]}{U'(-\infty)}.$$

(iii) $\theta^* > 0$, *that is a demand*, *if*:

$$\frac{U'[w(1+r)]}{U'(-\infty)} > -\frac{d}{u} > \frac{U'(+\infty)}{U'(-\infty)},$$

and then:

$$\theta^* = \frac{1}{u}\left\{U'^{-1}\left[-\frac{d}{u}U'(-\infty)\right] - w(1+r)\right\}.$$

When the initial budget varies, we get different patterns of the demand (supply) function according to the location of the expected risky return with respect to the riskfree rate for small p. First note that:

$$\frac{U'(-\infty)}{U'(+\infty)} > 1, \quad \frac{U'[w(1+r)]}{U'(+\infty)} > 1, \quad \frac{U'[w(1+r)]}{U'(-\infty)} < 1.$$

(1) If $u + d > 0$, i.e. if the expected risky return is larger than the riskfree rate, the limiting allocation is non negative, and is strictly positive if the initial budget is small enough:

$$w(1+r) < U'^{-1}\left[-\frac{d}{u}U'(-\infty)\right].$$

(2) If $u + d < 0$, i.e. if the expected risky return is smaller than the riskfree rate, the limiting allocation is non positive, and the investor has a strict short position in the risky asset, if the initial budget is sufficiently large:

$$w(1+r) > U'^{-1}\left[-\frac{d}{u}U'(+\infty)\right].$$

Thus the demand (or the supply) is a piecewise linear function of the initial budget.

2.3. Continuous infrequent extreme risks

We have considered above dichotomous "up" and "down" risks. The approach is easily extended to multivariate continuous risks by considering stochastic price movements "u" and "d" within regimes. More precisely the payoffs of the risky assets for the period $[t, t+1]$ are:

$$\begin{aligned} y_{j,t+1} = \frac{p_{j,t+1}}{p_{j,t}} - 1 &= r + u_{j,t}, \text{ with probability } 1 - p, \\ &= r + \frac{d_{j,t}}{p}, \text{ with probability } p, \end{aligned}$$

for $j = 1, \ldots, n$ where $u_t = (u_{1,t}, \ldots, u_{n,t})'$, $d_t = (d_{1,t}, \ldots, d_{n,t})'$ are independent random vectors with stationary distributions:

$$u_t \sim f_0(u), d_t \sim f_1(d), \text{(say)}.$$

For instance let us consider the univariate case. The optimization problem (2.4) becomes:

$$\max_\theta (1 - p)E_0U[w(1 + r) + \theta u] + pE_1U\left[w(1 + r) + \theta\frac{d}{p}\right],$$

where E_0 and E_1 denotes the expectations with respect to the distributions f_0 and f_1, respectively.

The values of the derivative of the objective function for $\theta = -\infty$ and $\theta = +\infty$ become, respectively:

$$\begin{aligned} &(1 - p)\{E_0(u\mathbb{1}_{u>0})U'(-\infty) + E_0(u\mathbb{1}_{u<0})U'(+\infty)\} \\ &\quad + \{E_1(d\mathbb{1}_{d<0})U'(+\infty) + E_1(d\mathbb{1}_{d>0})U'(-\infty)\}, \end{aligned}$$

and

$$\begin{aligned} &(1 - p)\{E_0(u\mathbb{1}_{u>0})U'(+\infty) + E_0(u\mathbb{1}_{u<0})U'(-\infty)\} \\ &\quad + E_1(d\mathbb{1}_{d<0})U'(-\infty) + E_1(d\mathbb{1}_{d>0})U'(+\infty)\}. \end{aligned}$$

Thus there is a finite solution (Proposition 2.1) if and only if.

$$\frac{U'(-\infty)}{U'(+\infty)} > -\frac{E_1(d\mathbb{1}_{d<0}) + E_0(u\mathbb{1}_{u<0})}{E_1(d\mathbb{1}_{d>0}) + E_0(u\mathbb{1}_{u>0})} > \frac{U'(+\infty)}{U'(-\infty)}.$$

The results are similar to Section 2.1 after replacing u by $E_1(d\mathbb{1}_{d>0}) + E_0(u\mathbb{1}_{u>0})$ and d by $E_1(d\mathbb{1}_{d<0}) + E_0(u\mathbb{1}_{u<0})$.

Finally note that in a multivariate framework infrequent extreme risks can exist for some portfolios and not for some others. As an example let us consider a mixture of gaussian distributions where $u_t \sim N[m_o, \Sigma_o]$, $d_t \sim N[0, \Sigma_1]$. In this framework the payoffs follow a mixture of gaussian distributions:

$$r(1, \ldots, 1)' + (1-p)N[m_o, \Sigma_o] + pN\left[0, \frac{\Sigma_1}{p^2}\right].$$

The assets are risky whenever $VY = (1-p)\Sigma_o + \frac{\Sigma_1}{p} + p(1-p)m_o m_o'$ has full rank. But the volatility matrix Σ_1 alone is not necessarily of full rank. In this case the portfolio allocations which do not include infrequent extreme risks correspond to the kernel of the volatility matrix Σ_1.

3. The LIRA utility functions

In this subsection we present the properties of the class of utility functions satisfying the condition of Proposition 2.2, that is $U'(-\infty) < \infty$. Since the existence of U' at $-\infty$ is equivalent to the integrability of the absolute risk aversion $-U''/U'$ at $-\infty$, this class will be called LIRA, for Left Integrable (Absolute) Risk Aversion. The integrability condition implies that the absolute risk aversion tends to zero at $-\infty$, that is the investor is risk neutral with respect to different very large losses.[2]

The class of LIRA utility functions does not include the standard utilities with Hyperbolic Absolute Risk Aversion (HARA) or Constant Absolute Risk Aversion (CARA). Indeed for HARA functions the demand functions are affine in w (Cass and Stiglitz [1970]). In particular a nondegenerate demand (supply) for extreme risks is not compatible with a mutual fund separation theorem.

3.1. A representation theorem

Proposition 3.1: *U is a LIRA utility function if and only if it can be written as*:

$$U(W) = b\int^{W} S(x)\,dx + aW + c, \quad a \geq 0, \quad b > 0,$$

where $S = 1 - G$ is a survivor function associated with a probability distribution on $\mathbb{R}$ with c.d.f. G.

Proof. We have:

$$U'(W) = U'(+\infty) + [U'(-\infty) - U'(+\infty)]\frac{U'(W) - U'(+\infty)}{U'(-\infty) - U'(+\infty)},$$

where

$$S(W) = \frac{U'(W) - U'(+\infty)}{U'(-\infty) - U'(+\infty)}$$

is a decreasing function, varying between 1 and 0. The result follows by integration. □

Therefore it is equivalent to select a subclass of LIRA utility functions, or to select a subfamily of probability distributions.

Note that the proof requires only the existence of the first order derivative of the utility function and its finite value at $-\infty$. Therefore the associated probability distribution can be either discrete, or continuous. When the distribution is continuous on a subinterval of $\mathbb{R}$, with a continuous density g, then the second order derivative of the utility function exists and is equal to $-bg$.

When $a = 0$ the link between the utility function and the probability distribution can alternatively be written in terms of the absolute risk aversion coefficient when the associated distribution is continuous. Indeed the risk aversion coefficient: $A(w) = -U''(w)/U'(w) = g(w)/S(w)$ coincides with the so-called hazard function of the probability distribution.

The previous representation may simplify the derivation of the individual demands. The utility function is equal to: $\int^{W} S(x)\,dx + \alpha W$, $\alpha \geq 0$, up to a positive multiplicative factor and the necessary condition of Proposition 2.1 becomes:

$$\frac{1+\alpha}{\alpha} > -\frac{d}{u} > \frac{\alpha}{1+\alpha}.$$

In the case of a positive risk premium for the risky asset (case $u + d > 0$), the limiting allocation is strictly positive if and only if:

$$w(1+r) < S^{-1}\left[\frac{-\alpha(u+d)-d}{u}\right],$$

and is equal to:

$$\theta^* = \frac{1}{u}\left\{S^{-1}\left[\frac{-\alpha(u+d)-d}{u}\right] - w(1+r)\right\}.$$

When $u + d < 0$, the limiting allocation is strictly negative, if and only if:

$$w(1+r) > S^{-1}\left[\frac{-\alpha(u+d)}{u}\right],$$

and is equal to:

$$\theta^* = \frac{1}{u}\left\{S^{-1}\left[\frac{-\alpha(u+d)}{u}\right] - w(1+r)\right\}.$$

3.2. *Interpretation in terms of option prices*

This interpretation is based on the representation of LIRA utility functions given below.

Proposition 3.2: *A LIRA utility function can be written as*:

$$U(W) = -bE_X[(W - X)^-] + aW + d, \quad a \geq 0, \quad b > 0,$$

where X denotes a random variable with distribution G.

Proof. We have:

$$\begin{aligned} E_X[(W - X)^-] &= -\int_W^{+\infty} (W - x) g(x)\, dx \\ &= \int_W^{+\infty} (W - x)\, dS(x) \\ &= [(W - x)S(x)]_W^{+\infty} + \int_W^{+\infty} S(x)\, dx \\ &= \text{const.} - \int^W S(x)\, dx. \end{aligned}$$

□

The decomposition formula given above has a simple interpretation, when the expected utility is considered:

$$\begin{aligned} EU(W) &= -bE_W E_X[(W - X)^-] + aEW + d \\ &= -bE_X E_W[(W - X)^-] + aEW + d, \\ EU(W) &= -bE_X \pi_X(W) + a\pi(W) + d, \end{aligned} \tag{3.1}$$

where $\pi(W)$ is the pure premium associated with the wealth W, and $\pi_X(W)$ the pure premium associated with the put based on the wealth W with strike X, that is the expected payoff evaluated with the historical probability (not with the risk neutral one). Alternatively $\pi_X(W)$ is the average excess of loss in a reinsurance context with retention X. Under these interpretations the probability G provides the weights on the strikes used to aggregate the put prices (resp. the excess loss).[3]

The above interpretation corresponds to standard practice of comparing risks of two assets. Indeed risk can be evaluated in two different ways. In the historical approach the investor observes the series of past payoffs and infers from this analysis an estimated conditional distribution of future payoffs. The first asset is considered more risky than the second one if its estimated conditional distribution stochastically dominates the estimated conditional distribution of the second asset. In the cross-sectional approach the investor compares the prices of the derivatives written on these assets. The first asset is more risky if the derivative prices (as function of the moneyness strike) are higher. The interpretation of the expected LIRA utility function corresponds to this cross-sectional approach, after replacing the risk neutral probability by the historical one. It provides a link between the historical and cross-sectional approaches.

The quantity $E_X \pi_X(W)$ is an absolute measure of risk related with some measures used on the market or discussed in the theoretical literature. Indeed this quantity can also be written as $E_P(W^-)$, where the probability P is the mixture of the historical probability drifted by X. Thus it can be interpreted as the expected loss associated with a scenario on future wealth, which modifies the historical probability by a random drift. The idea of expected loss computed for different scenarios is typically used on the Chicago Mercantile Exchange for determining the margin on futures. This system, called SPAN (Standard Portfolio Analysis of Risk), computes the maximum loss over 16 scenarios. The same type of computation has also been derived from an axiomatic definition of risk measures (see Artzner et al. [1996]). However the risk measure does not satisfy the axioms of homogeneity and subadditivity introduced in Kijima and Ohnishi [1993], Pedersen and Satchell [1998] and used by Artzner et al. [1999] to derive the so-called coherent risk measures. This is easily understood since all derivations performed above assume a given initial wealth, which does not allow for increasing the size of the portfolio.

Note finally that the restricted set of LIRA utility functions is sufficient to generate second order stochastic dominance. Indeed a random wealth W_1 is preferred to wealth W_o for any expected LIRA utility function if and only if: $EW_1 \geq EW_0$ and $\pi_X(W_1) \leq \pi_X(W_0), \forall X$. By a standard result (see e.g. Rothschild and Stiglitz [1970], Goovaerts, De Vylder, and Haezendonck [1984]), this condition is equivalent to the second order monotonic stochastic dominance (also called stop-loss ordering in insurance theory). Thus the set of LIRA utility functions generates second order stochastic dominance. By comparison it is known that the set of CARA utility functions does not generate this order. It is due to the difficulty to capture the extreme risk phenomena with the restricted class of CARA utilities.

3.3. Examples

Examples of LIRA utility functions can easily be derived by considering standard distributions for the strike. The distributions below are continuous, which ensures the existence of the first order derivative of the utility function. The second order derivative of the utility (and the risk aversion coefficient) exists if the associated probability density function is continuous.

Example 3.1 (Gaussian weights)**:** Let us consider the utility function:

$$U(W) = -bW\Phi(W) - b\varphi(W) + (a+b)W + c,$$

where φ and Φ are the pdf and the cdf of the standard normal. The marginal utility is:

$$U'(W) = a + b - b\Phi(W).$$

Thus $U'(-\infty) = a+b$ and the associated distribution is the standard normal distribution. The case of non standard normal distributions is easily deduced by applying an affine transformation to W.

It is interesting to note that a utility function with asymmetric properties concerning the preference for extreme risk can be derived from a symmetric distribution of the strike. This is a consequence of the asymmetric payoffs of put derivatives.

Example 3.2 (Double exponential weights)**:** The utility function is:

$$
\begin{aligned}
U(W) &= \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)^{-1} \frac{b}{\alpha_2^2}[1 - \exp -\alpha_2(W-d)] + aW + c, \quad \text{if } W > d,\\
&= -\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)^{-1} \frac{b}{\alpha_1^2} \exp \alpha_1(W-d) + \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)^{-1} \frac{b}{\alpha_1^2}\\
&\quad + b(W-d) + aW + c, \quad \text{if } W < d,
\end{aligned}
$$

where α_1, α_2 are positive parameters. The marginal utility is:

$$
\begin{aligned}
U'(W) &= \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)^{-1} \frac{b}{\alpha_2} \exp -\alpha_2(W-d) + a, \quad \text{if } W > d,\\
&= -\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)^{-1} \frac{b}{\alpha_1} \exp +\alpha_1(W-d) + b + a, \quad \text{if } W < d.
\end{aligned}
$$

It is a LIRA function associated with a double exponential distribution with survivor function:

$$
\begin{aligned}
S(W) &= \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)^{-1} \left[\frac{1}{\alpha_2} \exp -\alpha_2(W-d) \mathbb{1}_{W>d}\right.\\
&\quad \left. + \left[\frac{1}{\alpha_1} + \frac{1}{\alpha_2} - \frac{1}{\alpha_1} \exp \alpha_1(W-d)\right] \mathbb{1}_{W<d}\right].
\end{aligned}
$$

d is the mode of this distribution, whereas α_1 and α_2 are measures of left and right tail magnitudes, respectively. For $a = 0$ the utility function corresponds to a CARA utility function, when W is sufficiently large, and is modified for small values of W to ensure the left integrability of risk aversion; the absolute risk aversion is constant equal to α_2 for $W > d$, but decreases to zero with an exponential decay when W tends to $-\infty$.

Example 3.3 (Weights with left bounded support)**:** If we select a probability distribution with a left bounded support (γ, ∞), say, the associated utility functions are:

$$
\begin{aligned}
U(W) &= b \int_\gamma^W S(x)\,dx + aW + c\\
&= b(W-\gamma) + aW + c, \quad \text{if } W < \gamma,\\
&= b \int_\gamma^W S(x)\,dx + aW + c, \quad \text{if } W \geq \gamma,
\end{aligned}
$$

since $S(W) = 1$, if $W < \gamma$. The associated utility function is linear affine for W sufficiently small. The parameter γ represents a wealth target at which the pattern of risk aversion changes (see the discussion of this point in the survey by Fishburn and Kochenberger [1979], or in Pedersen [1999]).

For instance if the probability distribution is uniform on the interval $[\gamma, \delta]$, say, we get:

$$
\begin{aligned}
S(W) &= 1, \quad \text{if } W \leq \gamma, \\
&= \frac{\delta - W}{\delta - \gamma}, \quad \text{if } \gamma \leq W \leq \delta, \\
&= 0, \quad \text{if } W \geq \delta.
\end{aligned}
$$

The associated utility function is:

$$
\begin{aligned}
U(W) &= b(W - \gamma) + aW + c, \quad \text{if } W \leq \gamma \\
&= -\frac{b}{2}\frac{(\delta - W)^2}{\delta - \gamma} + \frac{b}{2}(\delta - \gamma) + aW + c, \quad \text{if } \gamma \leq W \leq \delta, \\
&= \frac{b}{2}(\delta - \gamma) + aW + c, \quad \text{if } W \geq \delta.
\end{aligned}
$$

It corresponds to a portion of quadratic function extended by two straight lines. The affine restriction on the right side ensures that the function is increasing, which is not naturally satisfied by a quadratic utility. The affine restriction on the left side ensures nondegenerate demand for extreme risks.

Of course the quadratic assumption can be weakened by considering the survivor function:

$$
\begin{aligned}
S(W) &= 1, \quad \text{if } W \leq \gamma, \\
&= \frac{(\delta - W)^\alpha}{(\delta - \gamma)^\alpha}, \quad \text{if } \gamma \leq W \leq \delta, \\
&= 0, \quad \text{if } W \geq \delta.
\end{aligned}
$$

The associated utility function becomes:

$$
\begin{aligned}
U(W) &= b(W - \gamma) + aW + c, \quad \text{if } W \leq \gamma, \\
&= -\frac{b(\delta - W)^{\alpha+1}}{(\alpha + 1)(\delta - \gamma)^\alpha} + \frac{b}{\alpha + 1}(\delta - \gamma) + aW + c, \quad \text{if } \gamma \leq W \leq \delta, \\
&= \frac{b}{(\alpha + 1)}(\delta - \gamma) + aW + c, \quad \text{if } W \geq \delta,
\end{aligned}
$$

which corresponds to an earlier suggestion by Fishburn [1974], or Pedersen [1999]. When α varies, the slopes of the two limiting straight lines stay the same, whereas the curvature of the nonlinear portion relying these lines is modified. In the limiting case $\alpha = 0$ the utility function reduces to a concave piecewise linear function with three regimes.

4. Equilibrium model

The LIRA utility functions can be used to get extensions of the standard CAPM equilibrium model. As usual there exists an infinity of equilibrium price distributions compatible with the rational expectation equilibrium restriction. In Section 4.1 we discuss the existence of an equilibrium distribution featuring binary infrequent extreme risk. Then in the next section we study the skewness properties of the equilibrium distributions. By considering a simple example, we show that there can coexist a gaussian equilibrium price distribution, which is in particular symmetric around its mean, and an associated equilibrium pricing kernel which is skewed.

4.1. The basic model

Let us assume a continuum of investors with the same utility function $U(W) = \int^W S(x)\,dx + \alpha W$, $\alpha \geq 0$, and a perfect knowledge of the distribution of the risky asset payoff and of the riskfree interest rate, but with different initial budgets $w^* = w(1+r)$, whose distribution is μ. We have seen in Section 2.2 that all these individual investors are on the same side of the market (for instance the demand side in case 1); this situation also occurs in the standard CAPM based on mean-variance behaviours (Sharpe [1964], Lintner [1965]). Let us consider the demand case, introduce an exogenous supply of the risky asset θ_S, say, and search for an equilibrium condition compatible with dichotomous infrequent extreme risks. This condition is:

$$\theta_S = \xi(u, d), \tag{4.1}$$

where

$$\xi(u, d) = \frac{1}{u} E_{w^*}\left[\left\{S^{-1}\left[\frac{-\alpha(u+d)-d}{u}\right] - w^*\right\} \mathbb{1}_{w^* < S^{-1}\left[\frac{-\alpha(u+d)-d}{u}\right]}\right],$$

is the aggregate demand function. Note that at equilibrium, we have necessarily $u + d > 0$.

In the context of infrequent extreme risks the aggregate demand cannot be interpreted as the demand of a representative investor with initial wealth Ew^*. Indeed, whenever the support of the distribution of initial budgets includes the limiting value $S^{-1}[\frac{-\alpha(u+d)-d}{u}]$, the aggregate demand does not depends on μ through Ew^* only.

It is important to note that the initial price of the risky asset has been normalized to one, and that the asset is characterized by u and d. In the standard equilibrium theory, the asset design is determined by a single parameter, usually the initial price. In our framework several parameterizations can be introduced. For instance we can consider that d (resp. u) is fixed and the value u (resp. d) is modified to reach the equilibrium. We can also change the up and down movements by a scale parameter $u = \lambda u_o$, $d = \lambda d_o$, where u_o, d_o are fixed and look for the equilibrium value of λ. The existence and properties of the equilibria are deduced from the properties of the aggregate demand function.

Proposition 4.1: *The aggregate demand function is*:

(i) *decreasing in* $-d$;

(ii) *homogenous of degree* -1 *with respect to* u, d;

(iii) *in particular there exists a unique equilibrium in either* $-d$, *or* λ.

Proof.

(i) The individual demand functions are increasing in d and the first property is an immediate consequence by integrating.

(ii) follows immediately.

(iii) let us consider the set of all possible values of ξ, when d varies. We know that: $1 > \frac{-d}{u} > \frac{\alpha}{1+\alpha}$ because of Proposition 2.1 and the condition for positive demand. Therefore:

$$1 > \frac{-\alpha(u+d)-d}{u} > 0.$$

Let us denote by $d_o = -u\frac{\alpha}{1+\alpha}$ and $d_1 = -u$ the values of d for which the two limiting values 0 and 1 are reached. We have $\xi(u, d_o) = +\infty$, $\xi(u, d_1) = 0$, and the image of ξ is the set of positive real numbers. We deduce the existence of an equilibrium in d and its uniqueness is a consequence of part (i). Moreover the existence and uniqueness of the equilibrium in λ immediately follows from $\xi(u_0\lambda, d_0\lambda) = \lambda^{-1}\xi(u_0, d_0)$. □

If we no longer assume that the price of the risky asset at t is one, but p_t, and if we assume that the payoff of the risky asset at $t+1$ is $(1+r+u)p_t$ with probability $1-p$ and $(1+r+\frac{d}{p})p_t$ with probability p, the aggregate demand is given by (4.1) in which u and d are replaced respectively by up_t and dp_t; therefore, the existence and the uniqueness of the equilibrium in p_t follows directly from the one in λ.

4.2. *Skewness of historical and risk neutral equilibrium price distributions*

For expository purpose let us discuss the skewness properties in a special framework of one investor, with initial wealth w and utility function associated with the point mass at η:

$$U(W) = \int^{W} S(x)\,dx + \alpha W,$$

where $S(W) = \mathbb{1}_{W<\eta}$.

We have:

$$\begin{aligned} U(W) &= \int_{\eta}^{W} S(x)\,dx + \alpha W \\ U(W) &= W - \eta + \alpha W, \quad \text{if } W \leq \eta, \\ &= \alpha W, \quad \text{if } W > \eta, \end{aligned}$$

and U is piecewise linear.

Denoting by p_{ot} and p_t the price of the riskless and risky assets at time t, the optimization problem is:

$$\begin{aligned} &\underset{\theta_0,\theta}{\text{Max}} \quad E_t U(\theta_0 p_{0,t+1} + \theta p_{t+1}) \\ &\text{s.t. :} \quad \theta_0 p_{ot} + \theta p_t = w, \end{aligned}$$

or, using $p_{0,t+1} = (1+r)p_{o,t}$ and $w^* = (1+r)w$

$$\underset{\theta}{\text{Max}}\, E_t U[\theta(p_{t+1} - (1+r)p_t) + w^*].$$

The first order condition of the optimization problem is:

$$E_t[(p_{t+1} - (1+r)p_t)\left[\mathbb{1}_{w^*+\theta(p_{t+1}-(1+r)p_t)<\eta} + \alpha\right] = 0. \tag{4.2}$$

Denoting $y_{t+1} = p_{t+1} - (1+r)p_t$, it is easily checked that, if $E_t y_{t+1} > 0$, y_{t+1} has a continuous conditional distribution and, if $E_t y_{t+1} \mathbb{1}_{y_{t+1<0}} + a E_t y_{t+1} < 0$, Eq. (4.2) has a unique positive solution in θ.

Thus for a positive exogenous supply θ_S the equilibrium condition becomes:

$$E_t \left\{ y_{t+1} \mathbb{1}_{y_{t+1} < \frac{\eta - w^*}{\theta_S}} + \alpha \right] = 0. \tag{4.3}$$

This equilibrium condition has a simple interpretation in terms of the risk return relationship, when $\alpha > 0$. Indeed we get:

$$E_t p_{t+1} = (1+r)p_t - \frac{1}{\alpha} E_t\left[(p_{t+1} - (1+r)p_t)\mathbb{1}_{p_{t+1}-(1+r)p_t < \frac{\eta - w^*}{\theta_S}}\right]. \tag{4.4}$$

The second component of the right hand side provides the risk premium, where the natural associated measure of risk is close to the standard Tail Value at Risk, introduced for management and control of extreme risks.

Let us now consider the equilibrium price distribution in the limiting case $\alpha = 0$. Condition (4.4) implies:

$$E_t\left[y_{t+1} \mathbb{1}_{y_{t+1} < \frac{\eta - w^*}{\theta_S}}\right] = 0, \tag{4.5}$$

with $m > 0$ and $\eta > w^*$. By assuming $y_{t+1} = \sigma z_{t+1}$, where $z_{t+1} \sim N[m, 1]$, it is easily checked that the function

$$\sigma \to E_t\left(z_{t+1} \mathbb{1}_{\sigma z_{t+1} < \frac{\eta - w^*}{\theta_S}}\right)$$

is decreasing and admits a zero. Therefore there exists a gaussian equilibrium price distribution and thus the choice of a LIRA utility function can be compatible with a symmetric price distribution.

However let us now consider the associated pricing distribution (risk neutral distribution). From (4.3) applied with $\alpha = 0$, we deduce:

$$p_t = \frac{1}{1+r} E_t\left[p_{t+1} \mathbb{1}_{p_{t+1}-(1+r)p_t < \frac{\eta - w^*}{\theta_S}}\right] \Big/ E_t\left[\mathbb{1}_{p_{t+1}-(1+r)p_t < \frac{\eta - w^*}{\theta_S}}\right],$$

and the expression of the stochastic discount factor for the period $(t, t+1)$:

$$M_{t,t+1} = \frac{1}{1+r} \mathbb{1}_{p_{t+1} < \frac{\eta - w^*}{\theta_S} + (1+r)p_t} / E_t\left(p_{t+1} < \frac{\eta - w^*}{\theta_S} + (1+r)p_t\right). \tag{4.6}$$

Thus the associated equilibrium risk neutral distributions are such that:

$$\begin{aligned} Q_t[p_{t+1} < k] &= E_t\left[(1+r)M_{t,t+1}\mathbb{1}_{p_{t+1}<k}\right] \\ &= P_t\left[p_{t+1} < k \mid p_{t+1} < \frac{\eta - w^*}{\theta_S} + (1+r)p_t\right], \end{aligned} \tag{4.7}$$

where Q_t (resp. P_t) denotes the risk neutral (resp. historical) distribution. Therefore the support of any risk neutral equilibrium distribution is included in $[-\infty, \frac{\eta - w^*}{\theta_S} + (1+r)p_t]$, and all these risk neutral distributions are left skewed, even if some historical equilibrium price distribution are not (as the gaussian distribution exhibited above).

5. Concluding remarks

We have introduced a notion of infrequent extreme risks, where a large loss on an asset occurs, but with a small probability. When investors want to include the corresponding risky assets in their portfolio, their demand (or supply) is non degenerate under some restrictions on their utility functions, leading to the so-called LIRA utility functions. A representation theorem allows for an interpretation of the associated expected utility as an average of pure premia of European puts. Finally, we discuss the existence and characteristics of the equilibria in the presence of infrequent extreme risks.

Appendix 1: Necessary and sufficient conditions for the existence of demands and supplies for extreme risks

(i) Proof of Proposition 2.2

Under the condition of Proposition 2.1, θ_p satisfies the first order condition:

$$u(1-p)U'[w(1+r) + \theta_p u] + dU'\left[w(1+r) + \theta_p \frac{d}{p}\right] = 0.$$

Let us denote by θ^* the positive finite limit of θ_p. The limit of $U'[w(1+r)+\theta_p \frac{d}{p}]$ exists, when p tends to zero, and is equal to $-\frac{u}{d}U'[w(1+r)+\theta^* u]$. The result follows, since $\lim_{p\to 0} U'[w(1+r)+\theta_p \frac{d}{p}] = U'(-\infty)$.

(ii) Proof of Proposition 2.3

When p tends to zero, the first order derivative of the expected utility function with respect to θ, $\theta \neq 0$, tends to:

$$\lim_{p\to 0} \left\{ u(1-p)U'[w(1+r)+\theta u] + dU'\left[w(1+r)+\theta\frac{d}{p}\right]\right\}$$
$$= uU'[w(1+r)+\theta u] + d[U'(-\infty)\mathbb{1}_{\theta>0} + U'(+\infty)\mathbb{1}_{\theta<0}].$$

This function is still decreasing with respect to θ, but is discontinuous at $\theta = 0$. It takes the following values:

θ	$-\infty$	0^-	0^+	$+\infty$
Value of the derivative	$uU'(-\infty)$ $+dU'(+\infty)$	$uU'[w(1+r)]$ $+dU'(+\infty)$	$uU'[w(1+r)]$ $+dU'(-\infty)$	$uU'(+\infty)$ $+dU'(-\infty)$

Let us distinguish several cases according to the location of $-\frac{d}{u}$ with respect to the ratios:

$$\frac{U'(-\infty)}{U'(+\infty)} > \frac{U'[w(1+r)]}{U'[+\infty]} > \frac{U'[w(1+r)]}{U'(-\infty)} > \frac{U'(+\infty)}{U'(-\infty)}.$$

(a) The condition of Proposition 2.1: $\frac{U'(-\infty)}{U'(+\infty)} > -\frac{d}{u} > \frac{U'(+\infty)}{U'(-\infty)}$ ensures that θ_p is in a compact interval for p sufficiently small and we deduce the convergence to a unique limit θ^* because of the concavity of the objective function.

(b) Moreover we have:

$$\theta^* < 0, \quad \text{if } \frac{U'(-\infty)}{U'(+\infty)} > -\frac{d}{u} > \frac{U'[w(1+r)]}{U'(+\infty)},$$
$$\theta^* = 0, \quad \text{if } \frac{U'[w(1+r)]}{U'(+\infty)} > -\frac{d}{u} > \frac{U'[w(1+r)]}{U'(-\infty)},$$
$$\theta^* > 0, \quad \text{if } \frac{U'[w(1+r)]}{U'(-\infty)} > -\frac{d}{u} > \frac{U'(+\infty)}{U'(-\infty)}.$$

Notes

1. Equivalently we can consider that Y is the relative asset price modification between t and $t+1$: $Y = p_{t+1}/p_t$, say.

2. It is interesting to compare this result with the discussion in Rabin [2000] on the links between the aversion for small and large risks. The fact that an expected utility maximizer, who "turns down gambles with loss $100 or gain $110, each with 50% probability, will also turn down 50–50 bets of losing $20,000 or gaining any sum", is due to the fixed 50–50 probabilities used to define the bets. When the potential loss increases, the loss probability has to decrease to stay on the expected utility indifference curve. This is what is done in the present paper.
3. The expected excess loss is a classical risk measure (see e.g. Stone [1973], Bawa and Lindenberg [1977]).

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